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Publication date:
2000

Citation for published version (APA):
Slikker, M., & Norde, H. W. (2000). *Incomplete Stable Structures in Symmetric Convex Games*. (CentER Discussion Paper; Vol. 2000-97). Operations research.

Center
for
Economic Research

No. 2000-97

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October 2000

ISSN 0924-7815

Incomplete stable structures in symmetric convex games

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September 26, 2000

Abstract

We study the model of link formation that was introduced by Aumann and Myerson (1988) and focus on symmetric convex games with transferable utilities. We show that with at most five players the full cooperation structure results according to a subgame perfect Nash equilibrium. Moreover, if the game is strictly convex then every subgame perfect Nash equilibrium results in a structure that is payoff equivalent to the full cooperation structure. Subsequently, we analyze a game with six players that is symmetric and strictly convex. We show that there exists a subgame Nash equilibrium that results in an incomplete structure in which two players are worse off than in the full cooperation structure, whereas four players are better off. Independent of the initial order any pair of players can end up being exploited.

Journal of Economic Literature classification numbers: C71, C72

Keywords: symmetric convex game, undirected graph, link formation, incomplete stable structure

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1 Introduction

In this paper the following question takes a central place: which cooperation structure between economic agents is likely to form? This question was first raised by Aumann and Myerson (1988). In their model the economic possibilities of the agents are described by a cooperative game, and the formation process of bilateral communication links by an extensive form game, in which pairs of players get the opportunity to form a link according to some specific rule of order, and the payoffs to the players are eventually determined by means of the Myerson value (cf. Myerson (1977)). Assuming that every player takes decisions to his own advantage, Aumann and Myerson (1988) already note that for superadditive games this process may lead to partial cooperation. In van den Nouweland (1993) it is conjectured that for convex games the complete cooperation structure can always be reached by letting the agents act according to some subgame perfect Nash equilibrium. Holzman[1] provided an example of a 5-person asymmetric convex game for which there exists an incomplete stable cooperation structure. Stability here means that if one considers the subgame of Aumann and Myerson's extensive form game with this incomplete cooperation structure as the starting point, then forming no additional links corresponds to a subgame perfect equilibrium. However, Holzman does not show that this particular structure can actually be reached. Feinberg (1998) presents a simple weighted majority with an incomplete stable structure.

In this paper we focus on symmetric convex games. It turns out that, even in the case that all players are symmetric, it is not only possible that an incomplete stable structure exists, but also that it will form according to some subgame perfect equilibrium. More precisely, we will show that in symmetric convex games with at most 5 players the full cooperation structure can be formed in equilibrium and in symmetric games with at most 5 players that are *strictly* convex all structures that can be formed in equilibrium result in the same payoffs as the full cooperation structure. Additionally, we analyze a 6-person game that is symmetric and strictly convex. This analysis illustrates that the arguments that suffice to show that the full cooperation structure can result in symmetric convex games with at most 5 players, cannot be extended to games with 6 (or more) players. Moreover, we show that in this 6-person game, according to the subgame perfect Nash equilibrium concept, structures can result in which two players receive strictly less than they would according to the full cooperation structure and four players receive strictly more. In fact, it turns out that any pair of players can be exploited, independent of the rule of order. We conclude with some remark on the influence of the order on the structures that result.

This paper is organized as follows. In section 2 some preliminaries about cooperative

[1] Unpublished, private communication.

games and cooperation structures are provided. The model of Aumann and Myerson is presented in section 3. Moreover, some results about the possible cooperation structures according to the subgame perfect Nash equilibrium concept are given for general allocation rules. In section 4 we focus on symmetric convex games with at most 5 players, whereas in section 5 the 6-person example is studied in detail.

2 Preliminaries

A cooperative game is a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ the characteristic function.

A cooperative game (N, v) is *superadditive* if for all $T_1 \subseteq N$ and all $T_2 \subseteq N \backslash T_1$ it holds that

$$v(T_1) + v(T_2) \leq v(T_1 \cup T_2). \tag{1}$$

A cooperative game (N, v) is *convex* if for all $i \in N$ and all $T_1 \subseteq T_2 \subseteq N$ with $i \in T_1$ it holds that

$$v(T_1) - v(T_1 \backslash \{i\}) \leq v(T_2) - v(T_2 \backslash \{i\}). \tag{2}$$

So, a game is convex if the marginal contribution of a player to any coalition is less than his marginal contribution to a larger coalition. A cooperative game is *strictly convex* if (2) holds with strict inequality for all $i \in N$ and all $T_1 \subset T_2 \subseteq N$ with $i \in T_1$.

Let N be a set of players and let $R \in 2^N \backslash \{\emptyset\}$. The *unanimity game* (N, u_R) is the game with $u_R(S) = 1$ if $R \subseteq S$ and $u_R(S) = 0$ otherwise (see Shapley (1953)). Every game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e. $v = \sum_{R \in 2^N \backslash \{\emptyset\}} \lambda_R(v) u_R$. If there is no ambiguity about the underlying game we will simply write $(\lambda_R)_{R \in 2^N \backslash \{\emptyset\}}$ instead of $(\lambda_R(v))_{R \in 2^N \backslash \{\emptyset\}}$. The *Shapley value* Φ of a game is now easily described by

$$\Phi_i(N, v) = \sum_{R \subseteq N:\ i \in R} \frac{\lambda_R}{|R|} \text{ for all } i \in N.$$

A (*communication*) *graph* is a pair (N, L) where the set of vertices N represents the set of players and the set of edges L represents the set of bilateral (communication) links. Two players i and j are directly connected iff $\{i, j\} \in L$. Two players i and j are connected (directly or indirectly) iff $i = j$ or there exists a path between players i and j. The notion of connectedness induces a partition of the player set into communication components, where two players are in the same *communication component* if and only if they are connected. The set of communication components will be denoted by N/L. The component $C \in N/L$ containing player $i \in N$ will be denoted by $C_i(L)$. Furthermore, denote the subgraph on the vertices in coalition $S \subseteq N$ by $(S, L(S))$, where

$L(S) = \{\{i,j\} \in L \mid \{i,j\} \subseteq S\}$, and the partition of S into communication components according to graph $(S, L(S))$ by S/L. Furthermore, define $L^N = \{\{i,j\} \mid \{i,j\} \subseteq N\}$.

Myerson (1977) studied communication situations (N, v, L) where (N, v) is a cooperative game and (N, L) a communication graph. He introduced the *graph-restricted game* (N, v^L), where

$$v^L(S) = \sum_{C \in S/L} v(C) \text{ for all } S \subseteq N.$$

So, a coalition is split into communication components and the value of this coalition in the graph-restricted game is then defined as the sum of the values of the communication components in the original game. The Shapley value of the game (N, v^L) is usually referred to as the *Myerson value* of communication situation (N, v, L). Notation:

$$\mu(N, v, L) = \Phi(N, v^L).$$

In Borm et al. (1992) another allocation rule for communication situation is studied, the *position value*. This value determines the payoff to the players in two steps. First, a value for each communication link is determined, as the Shapley value of a game where the original communication links are the players. Then, each player receives half of the value of all the links he is involved in. Recently, Hamiache (1999) introduced yet another allocation rule for communication situations.

The analysis in this paper concentrates on symmetric games. A game (N, v) is symmetric if there exist $v_1, v_2, \ldots, v_{|N|} \in \mathbb{R}$ such that $v(S) = v_{|S|}$ for all $S \in 2^N \backslash \{\emptyset\}$. So, in a symmetric game the value of a coalition only depends on its size.

3 A model of link formation

In this section we describe the model of link formation introduced by Aumann and Myerson (1988). Furthermore, we will introduce some notation dealing with subgame perfect Nash equilibria and derive some preliminary results.

Let (N, v) be a cooperative game with $|N| \geq 2$ and let γ be an allocation rule for communication situations. Let σ be an exogenously given order of pairs of players. Formally, $\sigma : L^N \to \left\{1, 2, \ldots, \binom{n}{2}\right\}$ is a bijection where $\sigma(\{i,j\}) = k$ denotes that pair $\{i,j\}$ is in position k. We will denote the link formation game in extensive form determined by cooperative game (N, v), allocation rule γ, and initial order σ by $\Delta^{lf}(N, v, \gamma, \sigma)$. The game starts with no links formed. The first pair of players according to σ gets the opportunity to form a link. This link is actually formed if both players agree on forming this link. If a link is formed, it cannot be broken in a further stage of the game. After a pair of players decided on whether or not to form a link, the next pair of players according to σ who did not form a link with each other yet, gets the opportunity to do so. After the last pair of players in the order has had the opportunity to form a link, the first pair of players in the order who did not form a link with each other yet, gets a new opportunity to form the link between them. The process stops when, after the last link that has been formed, all pairs of players who have not formed a link with each other yet, have had a final opportunity to do so and declined this offer. Throughout the process of link formation the entire history of acceptances and rejections is known to all players. This process results in a set of links, which represents in conjunction with the player set an undirected graph. We will denote this set of links by L. The payoffs to the players are then determined by the allocation rule, i.e., if (N, L) is formed player $i \in N$ receives

$$\gamma_i(N, v, L).$$

In the original model of Aumann and Myerson (1988) player i receives his Myerson value $\mu_i(N, v, L)$. We will mainly restrict ourselves to the Myerson value too. Borm (1990) studies several examples using the position value rather than the Myerson value.

Aumann and Myerson (1988) already argued that since the game of link formation is of perfect information it has subgame perfect Nash equilibria. Furthermore, they note that the order in which two players in a pair decide whether or not to form a link has no influence (on the outcome of the game). Either order leads to the same outcome as simultaneous choice.

Though decisions in a link formation game in extensive form are made by the players, we will, with a slight abuse of notation, sometimes refer to a decision of a link where we actually mean the decisions of the players in the (potential) link. Consider such a decision of a link and assume that strategies are fixed after the decision of this link. If both players (weakly) prefer to form the link then we call the choice of the link to form the link subgame perfect. Furthermore, if at least one player (weakly) prefers not to form the link then we call the choice of the link not to form the link subgame perfect. We remark that if links play subgame perfect then one can easily determine subgame perfect play of the players that results in the same outcome.

We are interested in subgame perfect Nash equilibria in the link formation game in extensive form. To analyze these equilibria we need to study subgames. Let γ be an allocation rule for communication situations and let (N, v) be a game in strategic form. A link formation game in extensive form in which the links in A have already been formed is denoted by $\Delta^{lf}(N, v, \gamma, \rho, A)$, with $A \subset L^N$ a set of links, and $\rho : L^N \backslash A \to \left\{1, 2, \ldots, \binom{n}{2} - |A|\right\}$ an order of the pairs of players who did not form a link with each other yet. If L is the set of links that have been formed in the game then player $i \in N$ receives $\gamma_i(N, v, L \cup A)$. Furthermore, note that $\Delta^{lf}(N, v, \gamma, \sigma, \emptyset) = \Delta^{lf}(N, v, \gamma, \sigma)$. We denote the set of subgame perfect Nash equilibria of $\Delta^{lf}(N, v, \gamma, \sigma)$ by SPNE$(\Delta^{lf}(N, v, \gamma, \sigma))$ and, similarly, the set of subgame perfect Nash equilibria of $\Delta^{lf}(N,v,\gamma,\rho,A)$ by SPNE$(\Delta^{lf}(N,v,\gamma,\rho,A))$. Note that $\Delta^{lf}(N,v,\gamma,\rho,A)$ is interesting in itself since it describes a process of link formation when some links have been formed already.

To describe every subgame of $\Delta^{lf}(N,v,\gamma,\sigma)$ we denote for all $A \subset L^N$, all orders $\rho : L^N\backslash A \to \left\{1,2,\ldots,\binom{n}{2} - |A|\right\}$, and all $k \in \{0,\ldots,\binom{n}{2} - |A| - 1\}$ the game $\Delta^{lf}(N,v,\gamma,\rho,A,k)$ representing a link formation game in extensive form in which the links in A have already been formed and, after the last link has been formed k pairs of players have had the opportunity to form a link and have refused to do so. So, the link $\{i,j\}$ with $\rho(\{i,j\}) = k+1$ is next to decide whether or not they want to form a link. Note that $\Delta^{lf}(N,v,\gamma,\rho,A,0) = \Delta^{lf}(N,v,\gamma,\rho,A)$.

Consider the link formation game $\Delta^{lf}(N,v,\gamma,\sigma)$. Let $A \subset L^N$ and $\{i,j\} \in A$. Denote by $\sigma_{A,\{i,j\}}$ the order restricted to $L^N\backslash A$ that results when the links in A have been formed and $\{i,j\} \in A$ is the link in A that has been formed last. Then $\Delta^{lf}(N,v,\gamma,\sigma_{A,\{i,j\}},A)$ is a subgame of $\Delta^{lf}(N,v,\gamma,\sigma)$. Furthermore, for all $k \in \{0,\ldots,\binom{n}{2} - |A| - 1\}$ we have that $\Delta^{lf}(N,v,\gamma,\sigma_{A,\{i,j\}},A,k)$ is a subgame of $\Delta^{lf}(N,v,\gamma,\sigma)$. In fact, any subgame of $\Delta^{lf}(N,v,\gamma,\sigma)$ can be described in this way. Finally, we note that $\Delta^{lf}(N,v,\gamma,\sigma_{A,\{i,j\}},A,k)$ does not uniquely determine a subgame, since the links in $A\backslash\{\{i,j\}\}$ can have been formed in several orders.

We will introduce some additional notation dealing with graphs that result according to subgame perfect Nash equilibria in a link formation game and its subgames. Firstly, we call a graph (N,L) a *perfect equilibrium graph* if there exists a subgame perfect Nash equilibrium that results in the graph (N,L). We denote the set of perfect equilibrium graphs in $\Delta^{lf}(N,v,\gamma,\sigma)$ by PEG(N,v,γ,σ). Similarly, we denote the set of graphs that result according to subgame perfect equilibria in a subgame $\Delta^{lf}(N,v,\gamma,\rho,A)$ by PEG(N,v,γ,ρ,A).

Let (N,L) be a graph and let ρ be an order of the links in $L^N\backslash L$. The graph (N,L) is called *ρ-stable* with respect to cooperative game (N,v) and allocation rule γ if $(N,L) \in$ PEG(N,v,γ,ρ,L). If there is no ambiguity about (N,v) and γ we will sometimes simply call such a graph *ρ-stable*. A graph (N,L) is called (*strongly*) *stable* with respect to cooperative game (N,v) and allocation rule γ, or simply stable, if it is ρ-stable for *all* orders ρ. For notational convenience, we call the full cooperation structure stable, though no associated subgame starting with this graph is defined.

Furthermore, a graph (N,L) is called *superstable* with respect to (N,v) and γ, or simply superstable, if (N,L) is the unique element of PEG(N,v,γ,ρ,L) for all orders ρ. For notational convenience the full cooperation structure is called superstable as well.

The following lemma deals with a condition on the allocation rule that ensures that the full cooperation structure can result according to a subgame perfect Nash equilibrium. This condition states that for all communication structures there exist two players who are not connected with each other directly and who both (weakly) prefer the full cooperation structure to the current structure.

Lemma 3.1 Let (N, v) be a cooperative game, let γ be an allocation rule for communication situations, and let σ be an order on the set of all pairs of players. If for all $L \subset L^N$ there exist $i, j \in N$ with $i \neq j$ and $\{i, j\} \notin L$ satisfying $\gamma_i(N, v, L) \leq \gamma_i(N, v, L^N)$ and $\gamma_j(N, v, L) \leq \gamma_j(N, v, L^N)$ then it holds that $(N, L^N) \in \text{PEG}(N, v, \gamma, \sigma)$.

Proof: Suppose that for all $L \subset L^N$ there exist $i, j \in N$ with $i \neq j$ and $\{i, j\} \notin L$ satisfying $\gamma_i(N, v, L) \leq \gamma_i(N, v, L^N)$ and $\gamma_j(N, v, L) \leq \gamma_j(N, v, L^N)$. We will, in fact, show that there exists a subgame perfect Nash equilibrium that restricted to any subgame $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{r,t\}}, A)$ with $A \subset L^N$ and $\{r, t\} \in A$ results in the full cooperation structure.

Consider the following strategy profile s. For any decision node let A denote the set of links that have been formed so far and let player i be a player who has to make a decision at this node. Let the strategy of player i be that he wants to form the proposed link if $\gamma_i(N, v, A) \leq \gamma_i(N, v, L^N)$. Since for all $L \subset L^N$ there exist $i, j \in N$ with $i \neq j$ and $\{i, j\} \notin L$ satisfying $\gamma_i(N, v, L) \leq \gamma_i(N, v, L^N)$ and $\gamma_j(N, v, L) \leq \gamma_j(N, v, L^N)$ it is obvious that this strategy profile results in the formation of the full cooperation structure, also if the strategy profile is restricted to some subgame $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{r,t\}}, A)$.

It remains to show that this strategy profile is a subgame perfect Nash equilibrium. By backward induction we only have to show that for each subgame the choice at the root according to s is subgame perfect assuming that the players choose according to s at all decision nodes of this subgame except for the choice at the root.

Consider an arbitrary subgame $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{r,t\}}, A, k)$. Suppose players i and j are the first pair of players who have to make a decision in $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{r,t\}}, A, k)$. We will distinguish between two cases: players i and j will form a link according to s and players i and j will not form a link according to s.

Firstly, assume that players i and j form a link according to s, i.e., $\gamma_i(N, v, A) \leq \gamma_i(N, v, L^N)$ and $\gamma_j(N, v, A) \leq \gamma_j(N, v, L^N)$. Suppose player i deviates from his choice at the root. If, according to s, no additional link is formed then (N, A) results. Alternatively, if according to s at least one link is formed then by construction of s the graph (N, L^N) will result eventually. In both cases, player i does not strictly improve his payoff, implying that his choice according to s at the root of $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{r,t\}}, A, k)$ is subgame perfect. By symmetry a similar argument holds for player j.

Secondly, assume that players i and j do not form a link according to s. Without loss of generality assume that $\gamma_i(N, v, A) > \gamma_i(N, v, L^N)$. Then according to s one of the graphs (N, A) and (N, L^N) results. Suppose player i deviates from his choice at the root. If this deviation has no effect on the formation of link $\{i, j\}$, i.e.,

$\gamma_j(N, v, A) > \gamma_j(N, v, L^N)$ as well, then this deviation has no effect on the graph that results. Alternatively, if the deviation of player i results in the formation of link $\{i, j\}$ then, by construction of s, the graph (N, L^N) will result eventually. In both cases player i does not strictly improve his payoff. Player j cannot influence the outcome of the game by changing his strategy at the root since player i prevents the formation of the link $\{i, j\}$ at the root of $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{r,t\}}, A, k)$ independent of the choice of player j.

We conclude that the full cooperation structure results according to a subgame perfect Nash equilibrium. □

Although the previous lemma provides a sufficient condition for the full cooperation structure to be supported by a subgame perfect Nash equilibrium, we cannot use this lemma to show that a specific graph is not supported by a subgame perfect Nash equilibrium. The following lemma deals with this issue.

Lemma 3.2 Let (N, v) be a cooperative game, let γ be an allocation rule for communication situations, and let σ be an order of all pairs of players. Let $\mathcal{L} \subseteq \mathrm{UG}^N$ be a set of undirected graphs with $(N, L^N) \in \mathcal{L}$. If for all $(N, L) \in \mathrm{UG}^N \backslash \mathcal{L}$ there exist $i, j \in N$ with $i \neq j$ and $\{i, j\} \notin L$ satisfying both $\gamma_i(N, v, L) < \gamma_i(N, v, L')$ and $\gamma_j(N, v, L) < \gamma_j(N, v, L')$ for all $L' \in \mathcal{L}$ with $L \subset L'$ then $\mathrm{PEG}(N, v, \gamma, \sigma) \subseteq \mathcal{L}$.

Proof: Assume that for all $(N, L) \in \mathrm{UG}^N \backslash \mathcal{L}$ there exist $i, j \in N$ with $i \neq j$ and $\{i, j\} \notin L$ satisfying both $\gamma_i(N, v, L) < \gamma_i(N, v, L')$ and $\gamma_j(N, v, L) < \gamma_j(N, v, L')$ for all $L' \in \mathcal{L}$ with $L \subset L'$. Furthermore, suppose that there exists a subgame perfect Nash equilibrium s that results in $(N, L) \notin \mathcal{L}$. Hence, there exists at least one subgame of $\Delta^{lf}(N, v, \gamma, \sigma)$, say $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{i,j\}}, A)$, with $(N, A) \notin \mathcal{L}$ and (N, A) being $\sigma_{A,\{i,j\}}$-stable, namely by taking $A = L$ and $\{i, j\}$ the last link that has formed according to s. Consider the set of subgames with this property and, subsequently, let $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{i,j\}}, A)$ be a subgame in this set with $|A|$ maximal. Then, by assumption there exist $r, t \in N$ with $r \neq t$ and $\{r, t\} \notin A$ satisfying both $\gamma_r(N, v, A) < \gamma_r(N, v, L')$ and $\gamma_t(N, v, A) < \gamma_t(N, v, L')$ for all $L' \in \mathcal{L}$ with $A \subset L'$. If r and t form a link then, by maximality of A and since $(N, L^N) \in \mathcal{L}$, it follows that some $L' \in \mathcal{L}$ will result with $\gamma_r(N, v, L') > \gamma_r(N, v, A)$ and $\gamma_t(N, v, L') > \gamma_t(N, v, A)$. So, at least one of the players r and t did not play subgame perfect according to the subgame perfect Nash equilibrium s' in $\Delta^{lf}(N, v, \gamma, \sigma_{A,\{i,j\}}, A)$ that resulted in (N, A). A contradiction. We conclude that the strategy profile s cannot be a subgame perfect Nash equilibrium. □

The following lemma is similar to lemma 3.2 starting with an initial set of links rather than with the empty graph. The proof is omitted since it is similar to the proof of lemma

3.2.

Lemma 3.3 Let (N, v) be a cooperative game, let γ be an allocation rule for communication situations, let $A \subset L^N$ be an initial set of links, and let σ be an order of all pairs of players in $L^N \backslash A$. Let $\mathcal{L} \subseteq \mathrm{UG}^N$ be a set of undirected graphs with $(N, L^N) \in \mathcal{L}$. If for all $(N, L) \in \mathrm{UG}^N \backslash \mathcal{L}$ with $A \subseteq L$ there exist $i, j \in N$ with $i \neq j$ and $\{i, j\} \notin L$ satisfying both $\gamma_i(N, v, L) < \gamma_i(N, v, L')$ and $\gamma_j(N, v, L) < \gamma_j(N, v, L')$ for all $L' \in \mathcal{L}$ with $L \subset L'$ then $\mathrm{PEG}(N, v, \gamma, \sigma, A) \subseteq \mathcal{L}$.

The lemmas above will play a prominent role in sections 4 and 5.

4 Symmetric convex games

The example of Holzman, to which we referred in the introduction, shows that it is not at all obvious that the full cooperation structure will result according to a subgame perfect Nash equilibrium if the underlying game is convex. The current section and section 5 deal with the issue whether more can be said if in addition to convexity we assume symmetry of the underlying game. In this section we restrict our analysis to symmetric convex games with at most 5 players. Section 5 deals with a 6-person symmetric convex game.

Restricting ourselves to symmetric games reduces the complexity of the analysis significantly, since we can restrict ourselves to *non-isomorphic* graphs. Two graphs (N_1, L_1) and (N_2, L_2) are *isomorphic* if there is a one-to-one correspondence between the vertices in N_1 and those in N_2 with the property that two vertices in N_1 are connected directly in (N_1, L_1) if and only if the corresponding vertices in N_2 are connected directly in (N_2, L_2). For example, graphs $(\{1, 2, 3\}, \{\{1, 2\}\})$ and $(\{1, 2, 3\}, \{\{1, 3\}\})$ are isomorphic. In fact, all graphs with three vertices and one link are isomorphic. In a communication situation with a symmetric convex game the payoff to a player only depends on his position in the graph and not on the specific labelling of the players. Hence, we can reduce the complexity of the analysis of symmetric games by calculating the payoff vectors for non-isomorphic graphs only.

The remainder of this section deals with subgame perfect Nash equilibria resulting in the full cooperation structure if the underlying game is symmetric and convex. Lemma 3.1 will play a prominent role in showing that the full cooperation structure is always supported by a subgame perfect Nash equilibrium in symmetric convex games with at most 5 players if the Myerson value is used to determine the payoffs. No condition on the initial order is required.

Before we present and prove the theorem we introduce some notation concerning symmetric games, which will be convenient in the analysis of symmetric convex games later on. Recall that a game (N, v) is symmetric if there exist numbers $v_1, \ldots, v_n$ such that $v(S) = v_{|S|}$ for all $S \in 2^N \backslash \{\emptyset\}$.

We will describe a basis for the class of symmetric games with player set N. For all $j \in \{1, \ldots, n\}$ define the cooperative game (N, w^j) by

$$w^j(S) = \begin{cases} 0 & \text{if } |S| \leq j-1; \\ |S|+1-j & \text{if } |S| \geq j. \end{cases}$$

Consider, for example, the 5-person cooperative game (N, w^3). This game is described by $w^3(S) = 0$ if $|S| \leq 2$ and $w^3(S) = |S| - 2$ if $|S| \geq 3$.

If (N, v) is not only symmetric, but also convex then by definition the numbers

$$\begin{aligned} \alpha_1 &= v_1, \\ \alpha_2 &= (v_2 - v_1) - (v_1 - 0), \\ \alpha_3 &= (v_3 - v_2) - (v_2 - v_1), \\ &\vdots \\ \alpha_n &= (v_n - v_{n-1}) - (v_{n-1} - v_{n-2}), \end{aligned}$$

are such that $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \geq 0, \ldots, \alpha_n \geq 0$. These inequalities are strict if (N, v) is strictly convex. Moreover, one easily verifies that $v_i = \sum_{j=1}^{i}(i+1-j)\alpha_j$ for every $i \in \{1, \ldots, n\}$.

Combining $v_i = \sum_{j=1}^{i}(i+1-j)\alpha_j$ with the definition of (N, w^j) we obtain the following decomposition of (N, v),

$$v = \sum_{j=1}^{n} \alpha_j w^j. \tag{3}$$

We already noted that for any symmetric convex game it holds that $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \geq 0, \ldots, \alpha_n \geq 0$. On the other hand, one easily verifies that for every $\alpha_1 \in \mathbb{R}$, $\alpha_2 \geq 0, \ldots, \alpha_n \geq 0$ the cooperative game (N, v), defined by (3), is a symmetric convex game. If (N, L) is a graph and v is defined by (3) then, by linearity of the Myerson value, we have

$$\mu_i(N, v, L) = \sum_{j=1}^{n} \alpha_j \mu_i(N, w^j, L),$$

for every $i \in \{1, \ldots, n\}$. Since the cooperative games (N, w^j), $j \in \{1, \ldots, n\}$, form a basis for the the set of symmetric games with player set N we can easily determine the Myerson value value of any communication situation (N, v, L) using the vector $(\mu_i(N, w^1, L), \ldots, (\mu_i(N, w^n, L))$.

We will provide a systematic list of Myerson values for symmetric games with at most 5 players. In order to do so we will denote graphs by a binary representation. Links are ordered in the following way:

$$\Big(\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3\}, \ldots, \{2,n\}, \{3,4\}, \ldots, \{n-1,n\}\Big).$$

A graph is denoted by a binary vector, where a 0 in the position of link $\{i,j\}$ represents that $\{i,j\}$ does not belong to the set of links, whereas a 1 in the position of link $\{i,j\}$ represents that $\{i,j\}$ is a link in the graph. So, $(1,1,0)$ represents graph $(\{1,2,3\},\{\{1,2\},\{1,3\}\})$.

First, consider a cooperative game (N,v) with one player. Clearly, this player receives $v_1=\alpha_1$ according to the Myerson value.

Secondly, consider a cooperative game (N,v) with two players. If (N,L) is the empty graph, i.e., the graph with binary representation (0), then both players receive $v_1 = 1\cdot\alpha_1+0\cdot\alpha_2$ according to the Myerson value, and hence their payoffs are identified with the vector $(1,0)$. If (N,L) is the graph connecting both players, i.e., the graph with binary representation (1), they will both receive $\frac{1}{2}v_2=\alpha_1+\frac{1}{2}\alpha_2$, and hence their payoffs are identified with the vector $(1,\frac{1}{2})$.

Now, consider symmetric convex games with 3 players. Since the payoff to a player, according to the Myerson value, only depends on the characteristics of the component he belongs to, it suffices to consider connected graphs. The payoffs to the players in the graphs with zero or one link already follow from the payoffs to the players for cooperative games with 1 and 2 players, since the largest component contains at most 2 players. Essentially, i.e., up to isomorphisms, there are only two connected graphs: those with 2 links and 3 links. The payoffs to the players in these two different graphs can be found in table 1, where the vectors for the ease of computation have been multiplied by the factor 60.

graph	player 1	player 2	player 3
(1,1,0)	(60,60,20)	(60,30,20)*	(60,30,20)*
(1,1,1)	(60,40,20)	(60,40,20)	(60,40,20)

Table 1: Payoffs in communication situations with symmetric 3-person games.

For example, the payoff to player 2 in graph $(\{1,2,3\},\{\{1,2\},\{1,3\}\})$ is given by

$$\begin{aligned}\mu_2(\{1,2,3\},v,\{\{1,2\},\{1,3\}\}) =&\frac{1}{60}(60\alpha_1+30\alpha_2+20\alpha_3)\\ =&v_1+\frac{1}{2}(v_2-2v_1)+\frac{1}{3}(v_3-2v_2+v_1)\\ =&\frac{1}{3}v_1-\frac{1}{6}v_2+\frac{1}{3}v_3.\end{aligned}$$

This follows from table 1, which states that for graph $(1,1,0)$ the payoff to player 2 is described by $(60,30,20)$. The stars (*) in table 1 and the following tables identify for each non-complete graph two players who are not connected with each other directly and who both (weakly) prefer the full cooperation structure to the current structure. Since the game (N, v) is convex we have $\alpha_i \geq 0$ for every $i \in \{2, \ldots, n\}$. So, we can put a star at every player in some non-complete graph if the corresponding vector is coordinate wise smaller than the vector associated with this player in the complete graph and the first coordinates of both vectors coincide.

In order to obtain a complete description of the payoffs for symmetric games with 4 players, we supplement table 1 with a table that describes the payoffs for connected graphs with 4 vertices. These payoffs can be found in table 2, where again the vectors have been multiplied by the factor 60.

graph	player 1	player 2	player 3	player 4
(1,1,1,0,0,0)	(60,90,45,15)	(60,30,25,15)	(60,30,25,15)*	(60,30,25,15)*
(1,1,0,0,1,0)	(60,60,40,15)	(60,60,40,15)	(60,30,20,15)*	(60,30,20,15)*
(1,1,1,1,0,0)	(60,70,45,15)	(60,40,25,15)	(60,40,25,15)*	(60,30,25,15)*
(1,1,0,0,1,1)	(60,45,30,15)*	(60,45,30,15)	(60,45,30,15)	(60,45,30,15)*
(1,1,1,1,1,0)	(60,50,30,15)	(60,50,30,15)	(60,40,30,15)*	(60,40,30,15)*
(1,1,1,1,1,1)	(60,45,30,15)	(60,45,30,15)	(60,45,30,15)	(60,45,30,15)

Table 2: Payoffs in communication situations with symmetric 4-person games.

Similarly, for symmetric convex games with 5 players we can use tables 1 and 2 if the graph is not connected. In table 3 we provide the payoffs for connected 5-person graphs only.

Using these three tables, we can prove the following theorem. We restrict ourselves to games with at least two players since no link formation game has been defined associated with a cooperative game with one player only.

Theorem 4.1 Let (N, v) be a symmetric convex game with at least 2 and at most 5 players. Then $(N, L^N) \in \text{PEG}(N, v, \mu, \sigma)$ for any order σ of the links in L^N.

Proof: If $|N| = 2$ both players will prefer to form a link and receive $\alpha_1 + \frac{1}{2}\alpha_2$ rather than not forming a link and receive α_1. From now on, assume that $|N| \geq 3$.

In view of lemma 3.1 it suffices to show that for all $L \subset L^N$ there exist $i, j \in N$ with $i \neq j$ and $\{i, j\} \notin L$ satisfying $\mu_i(N, v, L) \leq \mu_i(N, v, L^N)$ and $\mu_j(N, v, L) \leq \mu_j(N, v, L^N)$.

Let $L \subset L^N$. We will distinguish between two cases, $|N/L| > 1$ and $|N/L| = 1$. Firstly, assume that $|N/L| > 1$. Let $C_1, C_2 \in N/L$ with $C_1 \neq C_2$. By symmetry of (N, v) and, hence, of (N, v^{L^N}) it follows that

$$\mu_i(N, v, L^N) = \frac{v(N)}{|N|} \text{ for all } i \in N. \tag{4}$$

graph	player 1	player 2	player 3	player 4	player 5
(1,1,1,1,0,0,0,0,0,0)	(60,120,72,36,12)	(60,30,27,21,12)	(60,30,27,21,12)	(60,30,27,21,12)*	(60,30,27,21,12)*
(1,1,1,0,0,0,1,0,0,0)	(60,90,65,33,12)	(60,60,45,33,12)	(60,30,25,18,12)	(60,30,25,18,12)*	(60,30,20,18,12)*
(1,1,0,0,0,1,0,0,1,0)	(60,60,60,30,12)	(60,60,40,30,12)	(60,60,40,30,12)	(60,30,20,15,12)*	(60,30,20,15,12)*
(1,1,1,1,1,0,0,0,0,0)	(60,100,72,36,12)	(60,40,27,21,12)	(60,40,27,21,12)	(60,30,27,21,12)*	(60,30,27,21,12)*
(1,1,1,0,1,0,1,0,0,0)	(60,70,50,33,12)	(60,70,50,33,12)	(60,40,30,18,12)	(60,30,25,18,12)*	(60,30,25,18,12)*
(1,1,1,0,1,0,0,0,0,1)	(60,70,65,33,12)	(60,40,25,18,12)	(60,40,25,18,12)*	(60,60,45,33,12)	(60,30,20,18,12)*
(1,1,1,0,0,0,1,0,1,0)	(60,75,55,36,12)	(60,45,35,21,12)	(60,45,35,21,12)	(60,30,25,21,12)*	(60,45,30,21,12)*
(1,1,0,0,0,1,0,0,1,1)	(60,48,36,24,12)	(60,48,36,24,12)	(60,48,36,24,12)*	(60,48,36,24,12)*	(60,48,36,24,12)
(1,1,1,1,1,1,0,0,0,0)	(60,80,57,36,12)	(60,50,32,21,12)	(60,40,32,21,12)	(60,40,32,21,12)*	(60,30,27,21,12)*
(1,1,1,1,1,0,0,0,0,1)	(60,80,72,36,12)	(60,40,27,21,12)	(60,40,27,21,12)*	(60,40,27,21,12)	(60,40,27,21,12)*
(1,1,1,0,1,1,0,0,1,0)	(60,50,35,21,12)	(60,50,35,21,12)	(60,70,55,36,12)	(60,40,30,21,12)*	(60,30,25,21,12)*
(1,1,1,0,1,0,1,0,0,1)	(60,55,40,24,12)	(60,55,40,24,12)	(60,40,30,24,12)*	(60,45,35,24,12)	(60,45,35,24,12)*
(1,1,1,0,0,0,1,0,1,1)	(60,57,39,24,12)	(60,42,34,24,12)	(60,42,34,24,12)*	(60,42,34,24,12)*	(60,57,39,24,12)
(1,1,1,1,1,1,1,0,0,0)	(60,60,39,24,12)	(60,60,39,24,12)	(60,40,34,24,12)	(60,40,34,24,12)*	(60,40,34,24,12)*
(1,1,1,1,1,1,0,1,0,0)	(60,75,57,36,12)	(60,45,32,21,12)	(60,45,32,21,12)	(60,45,32,21,12)*	(60,30,27,21,12)*
(1,1,1,1,1,1,0,0,1,0)	(60,60,42,24,12)	(60,50,37,24,12)	(60,50,37,24,12)	(60,40,32,24,12)*	(60,40,32,24,12)*
(1,1,1,0,1,1,0,0,1,1)	(60,47,34,24,12)*	(60,47,34,24,12)	(60,52,39,24,12)	(60,52,39,24,12)	(60,42,34,24,12)*
(1,1,1,1,1,1,1,1,0,0)	(60,55,39,24,12)	(60,55,39,24,12)	(60,45,34,24,12)	(60,45,34,24,12)*	(60,40,34,24,12)*
(1,1,1,1,1,1,0,0,1,1)	(60,52,36,24,12)	(60,47,36,24,12)	(60,47,36,24,12)*	(60,47,36,24,12)*	(60,47,36,24,12)
(1,1,1,1,1,1,1,1,1,0)	(60,50,36,24,12)	(60,50,36,24,12)	(60,50,36,24,12)	(60,45,36,24,12)*	(60,45,36,24,12)*
(1,1,1,1,1,1,1,1,1,1)	(60,48,36,24,12)	(60,48,36,24,12)	(60,48,36,24,12)	(60,48,36,24,12)	(60,48,36,24,12)

Table 3: Payoffs in communication situations with symmetric 5-person games.

Convexity and symmetry of (N, v) imply that

$$\frac{v(N)}{|N|} \geq \frac{v(C)}{|C|} \text{ for all } C \in N/L. \tag{5}$$

By component efficiency of the Myerson value it follows that there exists $i \in C_1$ and $j \in C_2$ with $\mu_i(N, v, L) \leq \frac{v(C_1)}{|C_1|}$ and $\mu_j(N, v, L) \leq \frac{v(C_2)}{|C_2|}$. Combining this with (4) and (5) we find that $\mu_i(N, v, L) \leq \mu_i(N, v, L^N)$ and $\mu_j(N, v, L) \leq \mu_j(N, v, L^N)$.

Secondly, assume that $|N/L| = 1$. Then depending on whether $|N| = 3$, $|N| = 4$, or $|N| = 5$, a pair of players satisfying the required condition can be found, indicated by stars, in tables 1, 2, and 3, respectively.

This completes the proof. $\square$

Two structures are called *payoff equivalent* if they result in the same payoffs to the players. The following theorem states that in symmetric and strictly convex games with at most 5 players only structures can result that are payoff equivalent to the full cooperation structure.

Theorem 4.2 Let (N, v) be a game that is symmetric and strictly convex with at least 2 and at most 5 players. For any order σ it holds that if $(N, L) \in \text{PEG}(N, v, \mu, \sigma)$ then (N, L) is payoff equivalent to the full cooperation structure.

Proof: The theorem is obviously true if $|N| = 2$. From now on, assume that $|N| \geq 3$. It follows similar to the proof of theorem 4.1 that the conditions of lemma 3.2 rather than lemma 3.1 are satisfied, with $\mathcal{L}$ the set consisting of the graphs that are payoff equivalent to the complete graph. This completes the proof. $\square$

Consider a symmetric strictly convex game with at most 5 players. Then the set of graphs that are payoff equivalent to the full cooperation structure consists of the full cooperation structure itself and the graphs with a set of links that together form a cycle that traverses all points in the graph. Such a graph is usually called a *wheel*.

5 A 6-person symmetric convex game

This section is devoted entirely to the analysis of a specific 6-person symmetric convex game. The analysis of this game shows that we cannot extend the results of the previous section to games with more than 5 players in a straightforward manner. We will analyze the link formation games with this convex game as its underlying game and with all possible orders of the pairs of players. We will show that according to the subgame perfect Nash equilibrium concept a graph can result in which the players' payoffs are different from those in the full cooperation structure.

For notational convenience, in this section, we will sometimes denote an order σ by $(l_1, l_2, \ldots, l_k)$, meaning that for all $r \in \{1, \ldots, k\}$ it holds that link l_r is in position r according to σ, i.e., $\sigma(l_r) = r$. Furthermore, we will sometimes refer to a link $\{i, j\}$ by ij.

The game that is the main subject of study in this section is introduced in the following example.

Example 5.1 Let (N, v) be the 6-person symmetric convex game with player set $N =$

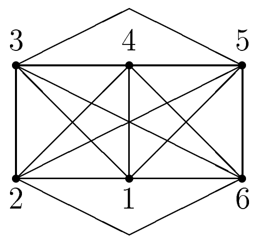
$\{1, 2, 3, 4, 5, 6\}$ and characteristic function v described by

$$v(T) = \begin{cases} 0 & \text{if } |T| \leq 1; \\ 60 & \text{if } |T| = 2; \\ 180 & \text{if } |T| = 3; \\ 360 & \text{if } |T| = 4; \\ 600 & \text{if } |T| = 5; \\ 1800 & \text{if } T = N. \end{cases} \tag{6}$$

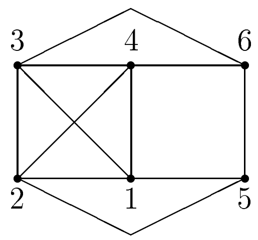
This characteristic function can alternatively be described by

$$v = 60 \sum_{i,j \in N : i \neq j} u_{\{i,j\}} + 900 u_N.$$

Though the game (N, w) with $w = \frac{1}{60}v$ has similar characteristics, we have chosen to analyze (N, v) to avoid non-integer payoffs to the players. Wilson and Watkins (1990) state that there exist 156 non-isomorphic graphs with 6 players (vertices). An overview of the payoffs to the players in communication situations (N, v, L), with (N, v) as described above and (N, L) any of the 156 non-isomorphic graphs, can be found in the appendix. We will refer to the graph with number i in the appendix by (N, L^i). Two of these graphs, namely (N, L^{146}) and (N, L^{156}), are represented in figure 1.



a: (N, L^{156})



b: (N, L^{146})

Figure 1: Graphs (N, L^{156}) and (N, L^{146}).

Graph (N, L^{156}) is the complete graph and graph (N, L^{146}) is the graph with all links except $\{1, 6\}$, $\{2, 6\}$, $\{3, 5\}$, and $\{4, 5\}$. The payoffs the players receive in the communication situations associated with these graphs are

$$\mu(N, v, L^{156}) = (300, 300, 300, 300, 300, 300)$$

and

$$\mu(N, v, L^{146}) = (301, 301, 301, 301, 298, 298),$$

which can be found in the appendix.

In graph (N, L^{146}) players 5 and 6 are the only players who are worse off than they are in the complete cooperation structure (N, L^{156}). Additionally, players 5 and 6 are connected with each other in (N, L^{146}). Hence, we cannot apply lemma 3.1 in an attempt to prove that the full cooperation structure results according to a subgame perfect Nash equilibrium. $\diamond$

Example 5.1 implies that the conditions of lemma 3.1 are not always satisfied for 6-person symmetric convex games. This example is in line with an unpublished example of R. Holzman, who analyzed a *non-symmetric* 5-person convex game that does not satisfy the conditions of lemma 3.1. By example 5.1 it follows that it is not possible to prove a generalization of theorem 4.2 to more than 5 players along the lines of the proof of theorem 4.2. However, example 5.1 does not imply that theorem 4.2 cannot be generalized to symmetric convex games with more than 5 players. This demands for a more extensive analysis of the game in example 5.1.

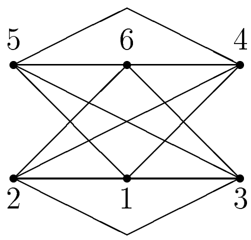
Example 5.2 Consider the game (N, v) of example 5.1. Furthermore, let $\mathcal{L}$ be the set of graphs that are payoff equivalent to the full cooperation structure or isomorphic to (N, L^{146}). Then, referring to the payoffs in the appendix, it follows that $\mathcal{L}$ is the set of graphs that are isomorphic to one of following graphs: (N, L^{156}), (N, L^{152}), (N, L^{146}), (N, L^{123}), (N, L^{122}), or (N, L^{54}). Graphs (N, L^{156}) and (N, L^{146}) were already represented in figure 1. The other four graphs can be found in figure 2.

We remark that though we consider a class of graphs with 6 non-isomorphic graphs only, many graphs belong this class. For example there exist 90 graphs that are, up to isomorphisms, the same as graph (N, L^{146}). This class of graphs will play an important role in the remainder of this section. $\diamond$

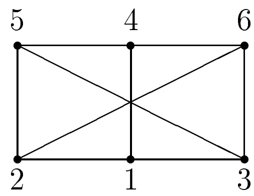
We will study to what extent we can narrow down the set of graphs that can form according to subgame perfect Nash equilibria in this example. We will use the following lemma, which states that for any subgame that starts right after a link has been formed and for the game in extensive form itself, it holds that according to the subgame perfect Nash equilibrium concept a graph payoff equivalent to the full cooperation structure or isomorphic to (N, L^{146}) will result.

Lemma 5.1 Let (N, v) be the 6-person game described by (6), let $\mathcal{L}$ be the set of graphs that are payoff equivalent to (N, L^N) or isomorphic to (N, L^{146}), let $L \subset L^N$, and let σ be any order of the links in $L^N \backslash L$. Then $\text{PEG}(N, v, \mu, \sigma, L) \subseteq \mathcal{L}$.

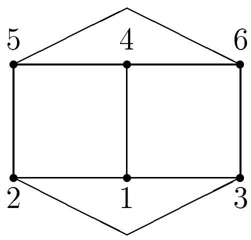
Proof: By the appendix it follows that for almost all (N, L) with $L \subseteq L^N$ and $(N, L) \notin \mathcal{L}$ there exists a pair of players $i, j \in N$ with $\{i, j\} \notin L$, $\mu_i(N, v, L) < 298$, and $\mu_j(N, v, L) <$



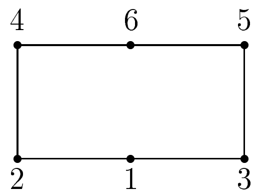
a: (N, L^{152})



b: (N, L^{123})



c: (N, L^{122})



d: (N, L^{54})

Figure 2: Graphs (N, L^{152}), (N, L^{123}), (N, L^{122}), and (N, L^{54}).

298. In fact, this holds for all graphs outside $\mathcal{L}$ except for the graphs isomorphic to one of the graphs indexed 151, 154, and 155. For none of these graphs it holds that the set of links of those graphs is a subset of L^{146} or of a graph isomorphic to L^{146}. For the graphs indexed 151, 154, and 155 there exists a pair of players $i, j \in N$ with $\{i, j\} \notin L$, $\mu_i(N, v, L) < 300$, and $\mu_j(N, v, L) < 300$. Now, lemma 3.3 completes the proof. $\square$

We will introduce some additional notation. Note that in any graph isomorphic to (N, L^{146}) two players are *exploited* by the others, i.e., they receive 298 only, whereas they would receive 300 in the full cooperation structure. Furthermore, note that any exploited player is connected with two not-exploited players besides the other exploited player. Denote the graph isomorphic to (N, L^{146}) with players i and j exploited and player i additionally connected to players r and t by $G^{i,j}_{r,t}$. Furthermore, let $G^{i,j} = \{G^{i,j}_{r,t} \mid r, t \in N \backslash \{i, j\}\}$ be the set of graphs isomorphic to (N, L^{146}) with players i and j exploited. We remark that $G^{i,j} = G^{j,i}$.

The following lemma will be used in the remainder of this section and deals with the stability of graphs isomorphic to graph (N, L^{146}).

Lemma 5.2 Let (N, v) be the 6-person game described by (6) and let $i, j, r, t \in N$ all be distinct. Then $G^{i,j}_{r,t}$ is superstable with respect to (N, v) and μ.

Proof: Without loss of generality assume that $i = 5$, $j = 6$, $r = 1$, and $t = 2$.

Consider $\Delta^{lf}(N, v, \mu, \sigma, G_{1,2}^{5,6})$ with σ an arbitrary order of the links $\{16, 26, 35, 45\}$. By lemma 5.1 it follows that once a link is formed in $\Delta^{lf}(N, v, \mu, \sigma, G_{1,2}^{5,6})$ the players will end up in the full cooperation structure. In all links that have not been formed yet a player (1, 2, 3, or 4) is involved who receives strictly more in the current structure, i.e., $G_{1,2}^{5,6}$, than in the full cooperation structure, namely 301 versus 300. Hence, this player strictly prefers the current structure to the full cooperation structure. By subgame perfectness we conclude first that the last pair in the order refuses if all other pairs have refused to form a link. Then, by backward induction, it follows that all other pairs will actually refuse to form a link as well. We conclude that no link will be formed in $\Delta^{lf}(N, v, \mu, \sigma, G_{1,2}^{5,6})$ and, hence, $G_{1,2}^{5,6}$ is superstable. $\square$

We can now prove the first main theorem of this section. This theorem states that for the link formation game associated with (N, v) of example 5.1 there exists an order of all pairs of players such that according to the subgame perfect Nash equilibrium concept a graph isomorphic to (N, L^{146}) can result.

Theorem 5.1 Let (N, v) be the 6-person game described by (6) and let $i, j, r, t \in N$ all be distinct. Then there exists an order σ of all pairs of players such that

$$G_{r,t}^{i,j} \in \text{PEG}(N, v, \mu, \sigma).$$

Proof: Without loss of generality let $i = 5$, $j = 6$, $r = 1$, and $t = 2$. Then

$$G_{1,2}^{5,6} = \Big\{56, 12, 13, 14, 23, 24, 34, 15, 36, 25, 46\Big\} \tag{7}$$

and

$$L^N \backslash G_{1,2}^{5,6} = \{16, 26, 35, 45\}. \tag{8}$$

We remark that $G_{1,2}^{5,6}$ is represented in figure 1 (b). For notational convenience we denote the links in $G_{1,2}^{5,6}$ by $l_1 = 56$, $l_2 = 12$, $\ldots$, $l_{10} = 25$, and $l_{11} = 46$, following the order in which the links were denoted in (7). Furthermore, we define $\Lambda_0 = \emptyset$ and

$$\Lambda_k = \{l_1, \ldots, l_k\}$$

for all $k \in \{1, 2 \ldots, 11\}$. Note that $\Lambda_{11} = G_{1,2}^{5,6}$. We will show by backward induction that $G_{1,2}^{5,6} = \Lambda_{11}$ results according to a subgame perfect Nash equilibrium in the game $\Delta^{lf}(N, v, \mu, \sigma)$ for the order of the links $\sigma = (16, 26, 35, 45, l_{11}, \ldots, l_1)$.

Define $\sigma_k = (16, 26, 35, 45, l_{11}, \ldots, l_{k+1})$ for all $k \in \{0, \ldots, 10\}$ and $\sigma_{11} = (16, 26, 35, 45)$. Note that $\sigma_0 = \sigma$. We will show that $G_{1,2}^{5,6}$ results according to a subgame perfect Nash equilibrium in $\Delta^{lf}(N, v, \mu, \sigma_k, \Lambda_k)$ for all $k \in \{0, \ldots, 11\}$. The proof will be by induction to $11 - k$.

Firstly, let $11-k=0$. Consider $\Delta^{lf}(N,v,\mu,\sigma_{11},\Lambda_{11}) = \Delta^{lf}(N,v,\mu,\sigma_{11},G^{5,6}_{1,2})$. Recall that $\sigma_{11} = (16,26,35,45)$, i.e., link 16 is first according to σ_{11}. By lemma 5.2 it follows that there exists a subgame perfect Nash equilibrium that results in $G^{5,6}_{1,2}$.

Secondly, let $k \in \{0,\ldots,10\}$ and suppose that $G^{5,6}_{1,2}$ results according to a subgame perfect Nash equilibrium in the game $\Delta^{lf}(N,v,\mu,\sigma_{k+1},\Lambda_{k+1})$.

Consider $\Delta^{lf}(N,v,\mu,\sigma_k,\Lambda_k)$. A subgame perfect Nash equilibrium s is described as follows. Firstly, we describe s in the subgames that start right after the formation of any additional link in $\Delta^{lf}(N,v,\mu,\sigma_k,\Lambda_k)$. For the subgame following the formation of l_{k+1} let s prescribe a subgame perfect Nash equilibrium that results in $G^{5,6}_{1,2}$, which is possible by the induction hypothesis. For the subgames following the formation of any $l \neq l_{k+1}$ fix any subgame perfect Nash equilibrium in this subgame.

It remains to describe for all $l \in L^N \backslash \Lambda_k$ the choice according to s in the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N,v,\mu,\sigma_k,\Lambda_k,\sigma_k(l)-1)$. Let $l \in L^N \backslash \Lambda_k$. Now, s prescribes that l is formed if $l = l_{k+1}$ and l is not formed if $l \neq l_{k+1}$.

It remains to show that s is indeed a subgame perfect Nash equilibrium. By construction it is a subgame perfect Nash equilibrium for all subgames following the formation of any link. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N,v,\mu,\sigma_k,\Lambda_k,\sigma_k(l)-1)$. We will distinguish between two cases.

Firstly, suppose $l = l_{k+1}$. By the payoffs in table 4 it follows that the players in l_{k+1} receive strictly less according to Λ_k than they would receive according to $G^{5,6}_{1,2}$. So, both players prefer to form link l_{k+1}, which implies that the choice of l according to s is subgame perfect.

Secondly, suppose l precedes l_{k+1} according to σ_k, not necessarily directly. If l deviates from s, i.e., forming l rather than not forming it, then both players in l will, according to lemma 5.1 receive at most 301, whereas at least one of them receives at least 301 according to $G^{5,6}_{1,2}$, which forms according to s. So, this player weakly prefers not to form l and, hence, the choice of this link according to s is subgame perfect.

We conclude that s is indeed a subgame perfect Nash equilibrium. Consequently, $G^{5,6}_{1,2}$ results according to a subgame perfect Nash equilibrium in the game $\Delta^{lf}(N,v,\mu,\sigma_k,\Lambda_k)$.

Since $\Delta^{lf}(N,v,\mu,\sigma_0,\Lambda_0) = \Delta^{lf}(N,v,\mu,\sigma)$ there exists a subgame perfect Nash equilibrium in $\Delta^{lf}(N,v,\mu,\sigma)$ that results in $G^{5,6}_{1,2}$. $\square$

Before we can prove the following theorem we need some additional notation. For all $G \subseteq L^N$ denote by $W(G)$ the set of links that have not been formed yet in which both

graph	isomorphic to	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	link
Λ_0	L^1	0	0	0	0	0	0	$l_1 = 56$
Λ_1	L^2	0	0	0	0	30	30	$l_2 = 12$
Λ_2	L^4	30	30	0	0	30	30	$l_3 = 13$
Λ_3	L^8	80	50	50	0	30	30	$l_4 = 14$
Λ_4	L^{13}	150	70	70	70	30	30	$l_5 = 23$
Λ_5	L^{25}	130	80	80	70	30	30	$l_6 = 24$
Λ_6	L^{42}	95	85	85	85	30	30	$l_7 = 34$
Λ_7	L^{68}	90	90	90	90	30	30	$l_8 = 15$
Λ_8	L^{88}	395	275	275	275	335	245	$l_9 = 36$
Λ_9	L^{113}	322	286	322	286	292	292	$l_{10} = 25$
Λ_{10}	L^{133}	301	301	314	290	307	287	$l_{11} = 46$
Λ_{11}	L^{146}	301	301	301	301	298	298	-

Table 4: Payoffs in Λ_k for all $k \in \{0, \ldots, 11\}$.

players receive strictly less than 301, i.e,

$$W(G) = \{l \in L^N \backslash G \mid \mu_a(N, v, G) < 301 \text{ for all } a \in l\}. \tag{9}$$

Furthermore, for all $i, j \in N$ with $i \neq j$ denote the graph with link $\{i, j\}$ and all links between the remaining players, by $F^{i,j}$, i.e.,

$$F^{i,j} = \{i, j\} \cup L^{N \backslash \{i,j\}}.$$

This graph, with $i = 5$ and $j = 6$, is represented in figure 3.

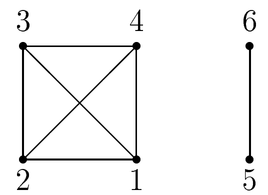


Figure 3: Graph $(N, F^{5,6})$, which coincides with (N, L^{68}).

In table 5 an overview of all non-isomorphic graphs G with $F^{5,6} \subseteq G \subseteq H$ for some $H \in G^{5,6}$ can be found, with associated $W(G)$. Using this table it can be checked that for all G with $F^{5,6} \subseteq G \subseteq H$ for some $H \in G^{5,6}$ and any l in $W(G)$ it holds that $F^{5,6} \subseteq G \cup \{l\} \subseteq H'$ for some $H' \in G^{5,6}$.

G	$W(G)$	$G \cup \{l\}$, $l \in W(G)$, is isomorphic to
(N, L^{68})	$\{15, 16, 25, 26, 35, 36, 45, 46\}$	(N, L^{88})
(N, L^{88})	$\{26, 36, 46\}$	(N, L^{113})
(N, L^{111})	$\{36, 46\}$	(N, L^{133})
(N, L^{113})	$\{25, 26, 45, 46\}$	(N, L^{133})
(N, L^{133})	$\{46\}$	(N, L^{146})
(N, L^{146})	$\emptyset$	-

Table 5: All non-isomorphic graphs G with $F^{5,6} \subseteq G \subseteq H$ for some $H \in G^{5,6}$.

Using table 5 we can prove the following theorem, which states that once a graph $F^{i,j}$ has been formed, then with certainty players i and j will be exploited according to any subgame perfect Nash equilibrium.

Theorem 5.2 Let (N, v) be the 6-person game described by (6) and let $i, j \in N$ with $i \neq j$. For all graphs G with $F^{i,j} \subseteq G \subseteq H$ for some $H \in G^{i,j}$ and every order σ of the links in $L^N \backslash G$ it holds that

$$\text{PEG}(N, v, \mu, \sigma, G) \subseteq G^{i,j}.$$

Proof: Without loss of generality assume that $i = 5$ and $j = 6$. The proof will be by induction to $11 - |G|$.

Firstly, let G be a graph with $11 - |G| = 0$, such that $F^{5,6} \subseteq G \subseteq H$ for some $H \in G^{5,6}$ and let σ be any order of the links in $L^N \backslash G$. Then clearly $G \in G^{5,6}$. By lemma 5.2 it follows that $\text{PEG}(N, v, \mu, \sigma, G) \subseteq G^{5,6}$.

Secondly, let $k \in \{0, 1, 2, 3\}$ and assume that for all graphs G with $11 - |G| = k$ and $F^{5,6} \subseteq G \subseteq H$ for some $H \in G^{5,6}$ and any order σ of the links in $L^N \backslash G$ it holds that $\text{PEG}(N, v, \mu, \sigma, G) \subseteq G^{5,6}$.

Let G be a graph with $11 - |G| = k + 1$ and $F^{5,6} \subseteq G \subseteq H$ for some $H \in G^{5,6}$ and let σ be an order of the links in $L^N \backslash G$. Let s be a subgame perfect Nash equilibrium in $\Delta^{lf}(N, v, \mu, \sigma, G)$. We will show that s results in a structure in $G^{5,6}$. In order to prove this we define

$$A = \{l \in L^N \backslash G \mid G \cup \{l\} \subseteq H \text{ for some } H \in G^{5,6}\};$$
$$B = \{l \in L^N \backslash G \mid l \notin A\}.$$

By the induction hypothesis we infer that s induces the formation of a structure in $G^{5,6}$ in every subgame that follows after the formation of a link in A. Moreover, s induces the formation of L^N in every subgame that follows after the formation of a link in B. This follows by lemma 5.1 and since L^N is the unique graph in $\mathcal{L}$ that contains all links of G and an arbitrary link of B.

In order to prove that s induces a structure in $G^{5,6}$ in $\Delta^{lf}(N, v, \mu, \sigma, G)$ we distinguish between two cases: (i) there exists an $l \in A$ such that s prescribes that l is formed (first) in $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$ and (ii) for all $l \in A$ it holds that s prescribes that l is not formed (first) in $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$.

Case (i): There exists an $l \in A$ such that s prescribes that l is formed (first) in $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$. Suppose s does not result in a structure in $G^{5,6}$. Then by lemma 5.1 it follows that s results in L^N. Hence, there exists a link $l' \in B$ that precedes l according to σ and l' chooses to form according to s in $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l') - 1)$. This would imply that both players in l' weakly prefer L^N to a structure in $G^{5,6}$. This cannot be true since l' contains at least one of the players in $\{1, 2, 3, 4\}$ who all receive 301 according to any $H \in G^{5,6}$.

Case (ii): For all $l \in A$ it holds that s prescribes that l is not formed (first) in subgame $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$. By lemma 5.1 we conclude that s induces structure L^N. Therefore, there exists $l \in B$ such that both players in l weakly prefer L^N to G. Stated differently, there exists a link $l \in L^N \backslash G$ such that $G \cup \{l\} \nsubseteq H$ for every $H \in G^{5,6}$ and $\mu_a(N, v, G) \leq \mu_a(N, v, L^N) = 300$ for both $a \in l$. This implies $l \in W(G)$. According to table 5 we get $G \cup \{l\} \subseteq H$ for some $H \in G^{5,6}$. A contradiction.

This completes the proof. $\square$

The following lemma is related to theorem 5.2, which implies that once a graph $F^{i,j}$ has been formed, a graph in $G^{i,j}$ *will* be reached according to a subgame perfect Nash equilibrium. The following lemma shows that if the link $\{i, j\}$ has been formed and only links have been formed that belong to $F^{i,j}$, then $G^{i,j}$ *can* be reached by a subgame perfect Nash equilibrium.

Lemma 5.3 Let (N, v) be the 6-person game described by (6) and let $i, j \in N$ with $i \neq j$. For all graphs G with $\{\{i, j\}\} \subseteq G \subseteq F^{i,j}$ and any order σ of the links in $L^N \backslash G$ it holds that $G^{i,j} \cap \text{PEG}(N, v, \mu, \sigma, G) \neq \emptyset$.

Proof: Without loss of generality assume that $i = 5$ and $j = 6$. The proof will be by induction to $7 - |G|$.

Firstly, let G be a graph with $7 - |G| = 0$ and $\{\{5, 6\}\} \subseteq G \subseteq F^{5,6}$, i.e., $G = F^{5,6}$, and let σ be any order of the links in $L^N \backslash G$. By theorem 5.2 it follows that $G^{5,6} \cap \text{PEG}(N, v, \mu, \sigma, G) = \text{PEG}(N, v, \mu, \sigma, G) \neq \emptyset$.

Secondly, let $k \in \{0,1,2,3,4,5\}$ and assume that for all graphs G with $7 - |G| = k$ and $\{\{5,6\}\} \subseteq G \subseteq F^{5,6}$ and any order σ of the links in $L^N \backslash G$ it holds that $G^{5,6} \cap \mathrm{PEG}(N, v, \mu, \sigma, G) \neq \emptyset$.

Let G be a graph with $7 - |G| = k+1$ and $\{\{5,6\}\} \subseteq G \subseteq F^{5,6}$ and let σ be an order of the links in $L^N \backslash G$. Let $\{a,b\}$ be the link that is last according to σ with $\{a,b\} \subseteq \{1,2,3,4\}$. A subgame perfect Nash equilibrium s is described as follows.

Firstly, we describe s in the subgames that start right after the formation of any additional link in $\Delta^{lf}(N, v, \mu, \sigma, G)$. For the subgame following the formation of $\{a,b\}$ let s prescribe a subgame perfect Nash equilibrium that results in $G^* \in G^{5,6}$, which is possible by the induction hypothesis, since $G \cup \{a,b\} \subseteq F^{5,6}$. For any subgame following the formation of an $l \neq \{a,b\}$, fix any subgame perfect Nash equilibrium in this subgame.

It remains to describe for all $l \in L^N \backslash G$ the choice according to s in the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$. Now, s prescribes that l is formed if $l = \{a,b\}$, l chooses subgame perfect if l is preceded by $\{a,b\}$ according to σ, and l is not formed if l precedes $\{a,b\}$ according to σ.

We will show that s is indeed a subgame perfect Nash equilibrium. By construction it is a subgame perfect Nash equilibrium for all subgames following the formation of any link. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$. We will distinguish between three cases.

Firstly, suppose l is preceded by $\{a,b\}$ according to σ. Then the choice of l is subgame perfect by construction.

Secondly, suppose $l = \{a,b\}$. Suppose l deviates from s, i.e., not forming $\{a,b\}$ rather than forming it. Then G results or l precedes a link according to σ that will be formed, in which case it follows by lemma 5.1 that a structure payoff equivalent to the full cooperation structure or isomorphic to (N, L^{146}) will result. In all cases the deviation does not strictly improve the payoffs of a or b since they will receive 301 if they play according to s. The fact that they both receive less according to G follows from the payoffs in the appendix. We conclude that players a and b weakly prefer to form the link and, hence, the choice of $\{a,b\}$ according to s is subgame perfect.

Finally, suppose l precedes $\{a,b\}$ according to σ, not necessarily directly. If l deviates from s, i.e., forming l rather than not forming it, then both players in l will, according to lemma 5.1 receive at most 301, whereas at least one of them receives 301 according to any $H \in G^{5,6}$, one of which is formed according to s. This player weakly prefers not to form l. Hence, the choice of l according to s is subgame perfect.

Consequently, s is a subgame perfect Nash equilibrium implying that

$$G^{5,6} \cap \mathrm{PEG}(N, v, \mu, \sigma, G) \neq \emptyset.$$

This completes the proof. $\square$

In the following lemma we consider a situation in which one link has been formed and the link that is last according to the order of the links that have not been formed yet has no player in common with the formed link. The lemma states that the players in the link that is last in the order can be exploited in a subgame perfect Nash equilibrium.

Lemma 5.4 Let (N, v) be the 6-person game described by (6). Let $i, j, k, l \in N$ all be distinct. Let $G = \{\{k, l\}\}$ and let σ be an order of the links in $L^N \backslash G$ with $\{i, j\}$ last. Then it holds that $G^{i,j} \cap \mathrm{PEG}(N, v, \mu, \sigma, G) \neq \emptyset$.

Proof: Without loss of generality, assume that $i = 5$, $j = 6$, $k = 3$, and $l = 4$. Then $G = \{\{3, 4\}\}$ has already been formed and $\{5, 6\}$ is last according to σ.

Consider $\Delta^{lf}(N, v, \mu, \sigma, G)$. A subgame perfect Nash equilibrium s is described as follows. Let $l \in L^N \backslash \{\{3, 4\}\}$. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$. Now, s prescribes that l is formed if $l = \{5, 6\}$ and l is not formed if $l \neq \{5, 6\}$. It remains to describe s in the subgames that start right after the formation of any additional link. For the subgame following the formation of $\{5, 6\}$ let s prescribe a subgame perfect Nash equilibrium that results in the formation of a specific $G^* \in G^{5,6}$, which is possible by lemma 5.3 since $\{\{5, 6\}\} \subseteq \{\{3, 4\}, \{5, 6\}\} \subseteq F^{5,6}$. For the subgames following the formation of any $l \neq \{5, 6\}$ fix any subgame perfect Nash equilibrium in this subgame.

It remains to show that s is indeed a subgame perfect Nash equilibrium. By construction it is a subgame perfect Nash equilibrium for all subgames following the formation of any link. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, G, \sigma(l) - 1)$. We will distinguish between two cases.

Firstly, suppose l precedes $\{5, 6\}$ according to σ, i.e., l is not last according to σ. If l deviates from s, i.e., forming l rather than not forming it, then both players in l will, according to lemma 5.1, receive at most 301. Since $l \neq \{5, 6\}$ it follows that $l \cap \{1, 2, 3, 4\} \neq \emptyset$ and, hence, at least one of the players in l receives 301 according to G^*. So, this player weakly prefers not to form the link and, hence, the choice of l according to s is subgame perfect.

Secondly, suppose $l = \{5, 6\}$. If l deviates from s, i.e., not forming $\{5, 6\}$ rather than forming it, then, graph $G = \{\{3, 4\}\}$ will result, since $\{5, 6\}$ is last according to σ. This deviation, hence, decreases the payoffs of both players. So, both players 5 and 6 prefer to form link l and, hence, the choice of $\{5, 6\}$ according to s is subgame perfect.

Consequently, s is a subgame perfect Nash equilibrium, implying that

$$G^{5,6} \cap \mathrm{PEG}(N, v, \mu, \sigma, G) \neq \emptyset.$$

This completes the proof. $\square$

The following lemma deals with a situation where two links have been formed already and there is a player who is involved in both links. The lemma implies that the player involved in both links, combined with any of the players he is connected with, can be exploited in a subgame perfect Nash equilibrium.

Lemma 5.5 Let (N, v) be the 6-person game described by (6). Let $i, j, k \in N$ all be distinct and let $G = \{\{i, j\}, \{j, k\}\}$. For any order σ of the links in $L^N \backslash G$ it holds that $G^{i,j} \cap \text{PEG}(N, v, \mu, \sigma, G) \neq \emptyset$.

Proof: Without loss of generality assume that $i = 5$, $j = 6$, and $k = 4$, i.e., $G = \{\{4, 6\}, \{5, 6\}\}$. Define $l_3 = 14$, $l_4 = 34$, $l_5 = 24$, $l_6 = 12$, $l_7 = 13$, and $l_8 = 23$. Furthermore, define $\Lambda_2 = G$ and $\Lambda_k = \{46, 56, l_3, \ldots, l_k\}$ for all $k \in \{3, \ldots, 8\}$. Note that Λ_8 satisfies $F^{5,6} \subseteq \Lambda_8 \subseteq G^{5,6}_{1,2}$. We will now prove that for all $k \in \{2, \ldots, 8\}$ and any order σ of the links in $L^N \backslash \Lambda_k$ it holds that $G^{5,6} \cap \text{PEG}(N, v, \mu, \sigma, \Lambda_k) \neq \emptyset$. The proof will be by induction to $8 - k$.

Firstly, let $8 - k = 0$, i.e., consider Λ_8. Let σ be an order of the links in $L^N \backslash \Lambda_8$. Since Λ_8 satisfies $F^{5,6} \subseteq \Lambda_8 \subseteq G^{5,6}_{1,2}$ it follows by theorem 5.2 that $G^{5,6} \cap \text{PEG}(N, v, \mu, \sigma, \Lambda_8) \neq \emptyset$.

Secondly, let $k \in \{3, 4, 5, 6, 7, 8\}$ and assume that for graph Λ_k and any order σ of the links in $L^N \backslash \Lambda_k$ it holds that $G^{5,6} \cap \text{PEG}(N, v, \mu, \sigma, \Lambda_k) \neq \emptyset$.

Consider $\Delta^{lf}(N, v, \mu, \sigma, \Lambda_{k-1})$. Construct a subgame perfect Nash equilibrium s similar to the strategy profile in the proof of lemma 5.3 except that the role of $\{a, b\}$ is played by l_k. Note that $\Lambda_k = \Lambda_{k-1} \cup \{l_k\}$. By table 6 it follows that both players in l_k prefer any $H \in G^{5,6}$ to structure Λ_{k-1}.

graph	isomorphic to	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	link
Λ_2	L^3	0	0	0	50	50	80	$l_3 = 14$
Λ_3	L^7	65	0	0	115	65	115	$l_4 = 34$
Λ_4	L^{12}	85	0	85	200	80	150	$l_5 = 24$
Λ_5	L^{21}	255	255	255	455	245	335	$l_6 = 12$
Λ_6	L^{38}	265	265	255	435	245	335	$l_7 = 13$
Λ_7	L^{62}	280	270	270	400	245	335	$l_8 = 23$
Λ_8	L^{88}	275	275	275	395	245	335	-

Table 6: Payoffs in Λ_k for all $k \in \{2, \ldots, 8\}$.

Now, it follows similar to the proof of lemma 5.3 that s is a subgame perfect Nash equilibrium, implying that

$$G^{5,6} \cap \mathrm{PEG}(N, v, \mu, \sigma, \Lambda_{k-1}) \neq \emptyset.$$

This completes the proof. □

Note that the proof provides a path of graphs such that if we consider the link formation game starting with this graph then a graph with two specific players exploited can result according to a subgame perfect Nash equilibrium.

The following lemma states that if one link has been formed and this link has one player in common with the pair of players that is last according to the order that results right after the formation of the link, then the pair of players that is last according to the order that results can be exploited according to a subgame perfect Nash equilibrium. The proof of this lemma has been omitted since it follows similar to the proof of lemma 5.4 using lemma 5.5 rather than lemma 5.3.

Lemma 5.6 Let (N, v) be the 6-person game described by (6). Let $i, j, k \in N$ all be distinct. Let $G = \{\{j, k\}\}$ and let σ be an order of the links in $L^N \backslash G$ with $\{i, j\}$ last. Then it holds that $G^{i,j} \cap \mathrm{PEG}(N, v, \mu, \sigma, G) \neq \emptyset$.

Using the lemmas above we can prove the last theorem of this section.

Theorem 5.3 Let (N, v) be the 6-person game described by (6). Let σ be any order of the links in L^N. For all $i, j \in N$ with $i \neq j$ it holds that $G^{i,j} \cap \mathrm{PEG}(N, v, \mu, \sigma) \neq \emptyset$.

Proof: Let $i, j \in N$ with $i \neq j$. Without loss of generality assume that $i = 5$ and $j = 6$. We will distinguish between two cases: (i) $\{5, 6\}$ is last according to σ and (ii) $\{5, 6\}$ is not last according to σ.

Case (i): $\{5, 6\}$ is last according to σ. A subgame perfect Nash equilibrium s is described as follows. Let $l \in L^N$. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, \emptyset, \sigma(l) - 1)$. Now, s prescribes that l is formed if $l = \{5, 6\}$ and l is not formed if $l \neq \{5, 6\}$. It remains to describe s in the subgames that start right after the formation of any additional link. For the subgame following the formation of $\{5, 6\}$ let s prescribe a subgame perfect Nash equilibrium that results in a specific $G^* \in G^{5,6}$, which is possible by lemma 5.3. For the subgames following the formation of any $l \neq \{5, 6\}$ fix any subgame perfect Nash equilibrium in this subgame.

It remains to show that s is indeed a subgame perfect Nash equilibrium. By construction it is a subgame perfect Nash equilibrium for all subgames following the formation of any link. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, \emptyset, \sigma(l) - 1)$. We will distinguish between two cases.

Firstly, suppose $l \neq \{5, 6\}$. If l deviates from s, i.e., forming l rather than not forming it, then both players in l will, according to lemma 5.1, receive at most 301. Since $l \neq \{5, 6\}$ it follows that $l \cap \{1, 2, 3, 4\} \neq \emptyset$ and, hence, at least one of the players in l receives 301 according to G^*. So, this player weakly prefers not to form link l, which implies that the choice of l according to s is subgame perfect.

Secondly, suppose $l = \{5, 6\}$. If l deviates from s, i.e., not forming $\{5, 6\}$ rather than forming it, then the empty graph will result. This deviation, hence, decreases the payoffs of both players. So, both players 5 and 6 prefer to form link l and, hence, the choice of $\{5, 6\}$ according to s is subgame perfect.

Consequently, s is a subgame perfect Nash equilibrium, implying that

$$G^{5,6} \cap \mathrm{PEG}(N, v, \mu, \sigma) \neq \emptyset.$$

Case (ii): $\{5, 6\}$ is not last according to σ. A subgame perfect Nash equilibrium s is described as follows. Let $l \in L^N$. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, \emptyset, \sigma(l) - 1)$. Now, s prescribes that l is formed if l is preceded by $\{5, 6\}$ according to σ or $l = \{5, 6\}$, and l is not formed if l precedes $\{5, 6\}$. It remains to describe s in the subgames that start right after the formation of any additional link. For the subgame following the formation of a link l that was preceded by $\{5, 6\}$ according to σ let s prescribe a subgame perfect Nash equilibrium that results in $H \in G^{a,b}$, where a, b are such that $\{a, b\}$ directly precedes l according to σ, i.e., $\sigma(\{a, b\}) = \sigma(l) - 1$. This is possible by lemmas 5.4 and 5.6. For the subgame following the formation of $\{5, 6\}$ let s prescribe a subgame perfect Nash equilibrium that results in a specific $G^* \in G^{5,6}$, which is possible by lemma 5.3. For the subgames following the formation of any l that precedes $\{5, 6\}$ according to σ fix any subgame perfect Nash equilibrium in this subgame.

It remains to show that s is indeed a subgame perfect Nash equilibrium. By construction it is a subgame perfect Nash equilibrium for all subgames following the formation of any link. Consider the decision node where link l has to make a decision and no links have been formed so far, i.e., the root of $\Delta^{lf}(N, v, \mu, \sigma, \emptyset, \sigma(l) - 1)$. We will distinguish between three cases.

Firstly, suppose l is preceded by $\{5, 6\}$ according to σ. Suppose l deviates from s, i.e., not forming l rather than forming it. If l is last according to σ then after the deviation the empty graph will result, decreasing the payoffs of both players in l. If l is not last according to σ then the link that follows l according to σ will be formed. In the subgame following the formation of this link it follows that after the deviation, both players in l

will receive 298. This deviation does not strictly improve the payoff of any of the players in l, since originally they would both receive at least 298. In both cases at least one of the players in l weakly prefers not to form l and, hence, the choice of l according to s is subgame perfect.

Secondly, suppose $l = \{5,6\}$. Similar to the analysis for the links that are preceded by $\{5,6\}$ it follows that a deviation of $\{5,6\}$ results in both players 5 and 6 receiving 298. These payoffs coincide with the payoffs players 5 and 6 receive originally. So, both players 5 and 6 weakly prefer to form link $\{5,6\}$ and, hence, the choice of $\{5,6\}$ according to s is subgame perfect.

Thirdly, suppose l precedes $\{5,6\}$ according to σ. If l deviates from s, i.e., forming l rather than not forming it, then both players in l will, according to lemma 5.1 receive at most 301. Since $l \neq \{5,6\}$ it follows that $l \cap \{1,2,3,4\} \neq \emptyset$ and, hence, at least one of the players in l receives 301 according to G^*. This player weakly prefers not to form l and, hence, the choice of this link according to s is subgame perfect.

Consequently, s is a subgame perfect Nash equilibrium, implying that

$$G^{5,6} \cap \mathrm{PEG}(N, v, \mu, \sigma) \neq \emptyset.$$

This completes the proof. $\square$

Theorem 5.3 provides a result for any initial order of the links. Hence, one might suspect that starting with an arbitrary set of links, the resulting set of graphs or payoffs are independent of the order of the links. An analysis that deals with this issue and that seems promising at first sight is to determine for the link formation games starting with each of the 156 non-isomorphic graphs and some order of the links that have not been formed whether a structure payoff equivalent to the full cooperation structure or isomorphic to (N, L^{146}) will result. Working our way back, starting with graph (N, L^{156}) this works well for graphs with at least 10 links. The first problems arise in graphs isomorphic to (N, L^{110}). These problems are illustrated in the following example.

Example 5.3 Consider graph (N, L^{110}), which is represented in figure 4. Only one structure isomorphic to (N, L^{146}) contains all links of graph (N, L^{110}), namely $G^{5,6}_{1,2}$. Additionally, it contains links 46 and 56. In communication situation (N, v, L^{110}), where v is the characteristic function of example 5.1, the payoffs the players receive according to the Myerson value equal

$$\mu(N, v, L^{110}) = (303, 303, 373, 285, 278, 258).$$

Obviously, there are player-specific preferences over the structures. Part of these preferences, those over structures (N, L^{110}), $G^{5,6}_{1,2}$, and full cooperation structure (N, L^{156}),

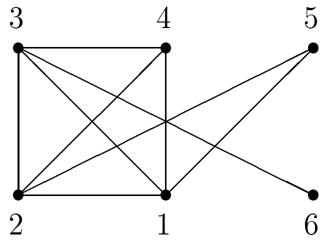


Figure 4: Graph (N, L^{110}).

are displayed in table 7.[2]

players	part of preference over structures
1,2,3	$(N, L^{110}) \succ G^{5,6}_{1,2} \succ (N, L^{156})$
4	$G^{5,6}_{1,2} \succ (N, L^{156}) \succ (N, L^{110})$
5,6	$(N, L^{156}) \succ G^{5,6}_{1,2} \succ (N, L^{110})$

Table 7: Part of preferences of the players

Consider a link formation game in extensive form starting with initial set of links L^{110} and some order of the missing links. We will analyze the structures that result according to subgame perfect Nash equilibria in a rather informal way. If possible, players 1, 2, and 3 would like to avoid the formation of an additional link. Hence, the only links that are relevant in the analysis are links 45, 46, and 56. It can be checked that if one of links 46 and 56 forms first the players end up in structure $G^{5,6}_{1,2}$. If any of the other links, i.e., link 45 or one of the links in which some of players 1, 2, and 3 are involved, forms first, then the full cooperation structure (N, L^{156}) will result.

We will argue that the order of links 45, 46, and 56 determines the cooperation structure that results. First, suppose 45 is last of those three links. Consider the decisions of the links if no links have been formed so far according to any subgame perfect Nash equilibrium. The links that are after 45 according to the order will, if all links before them refuse to form, choose not to form as well, since at least one of players 1, 2, and 3 is involved in any of these links and this player prefers (N, L^{110}) to (N, L^{156}). Link 45 will form since both players in this link prefer (N, L^{156}) to (N, L^{110}). Finally, suppose a link before 45 according to the order chooses form and a structure different from the full cooperation structure results. Obviously, this other structure is $G^{5,6}_{1,2}$ and the link under

[2]By $a \succ b$ we denote a player prefers a to b.

consideration has to be one of the links 46 and 56. Since player 6 prefers (N, L^{156}) to $G^{5,6}_{1,2}$ this decision of this link cannot be subgame perfect. We conclude that each subgame perfect Nash equilibrium results in full cooperation structure (N, L^{156}).

Secondly, with a similar analysis, we conclude that if 46 or 56 is last according to the order of the links 45, 46, and 56 then $G^{5,6}_{1,2}$ results according to any subgame perfect Nash equilibrium.

The dependence on the order of the links makes it hard to continue to identify the structures that result starting with graphs that are contained in a graph isomorphic to (N, L^{110}). $\diamond$

This section has been devoted completely to the analysis of a 6-person symmetric convex game, the game described by (6). More specifically, we analyzed the associated link formation games in extensive form. Two main conclusions can be drawn from this analysis. First, cf. theorem 5.2, once a 4-person coalition and a 2-person coalition have been formed, i.e., the full cooperation structure within those coalitions and no other links have been formed, then the players in the 2-person coalition will eventually be exploited. Secondly, cf. theorem 5.3, starting with no links formed, any order of the links can result in any pair of players being exploited. However, it is still unknown whether all subgame perfect Nash equilibria result in two players being exploited. Stated differently, the question

> "Does a subgame perfect Nash equilibrium that results in a structure payoff equivalent to the full cooperation structure exist?"

remains unanswered.

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Appendix: Non-isomorphic graphs with 6 players

This appendix deals with payoffs in communication situations with (N, v) of example 5.1 as the underlying game. Tables 8 through 13 provide an overview of these payoffs for all 156 non-isomorphic graphs with 6 players according to the Myerson value. Recall the binary representation of a graph from section 4.

number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
1	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	(0,	0,	0,	0,	0,	0)
2	(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	(30,	30,	0,	0,	0,	0)
3	(1,1,0,0,0,0,0,0,0,0,0,0,0,0,0)	(80,	50,	50,	0,	0,	0)
4	(1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)	(30,	30,	30,	30,	0,	0)
5	(1,1,1,0,0,0,0,0,0,0,0,0,0,0,0)	(150,	70,	70,	70,	0,	0)
6	(1,1,0,0,0,1,0,0,0,0,0,0,0,0,0)	(60,	60,	60,	0,	0,	0)
7	(1,1,0,0,0,0,1,0,0,0,0,0,0,0,0)	(115,	115,	65,	65,	0,	0)
8	(1,1,0,0,0,0,0,0,0,0,0,0,1,0,0)	(80,	50,	50,	30,	30,	0)
9	(1,0,0,0,0,0,0,0,0,1,0,0,0,0,1)	(30,	30,	30,	30,	30,	30)
10	(1,1,1,1,0,0,0,0,0,0,0,0,0,0,0)	(240,	90,	90,	90,	90,	0)
11	(1,1,1,0,0,1,0,0,0,0,0,0,0,0,0)	(130,	80,	80,	70,	0,	0)

Table 8: Payoffs according to the Myerson value for the game of example 5.1 and all non-isomorphic graphs, part 1.

number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
12	(1,1,1,0,0,0,0,1,0,0,0,0,0,0,0)	(200,	150,	85,	85,	80,	0)
13	(1,1,1,0,0,0,0,0,0,0,0,0,0,0,1)	(150,	70,	70,	70,	30,	30)
14	(1,1,0,0,0,1,0,0,0,0,0,0,1,0,0)	(60,	60,	60,	30,	30,	0)
15	(1,1,0,0,0,0,1,0,0,1,0,0,0,0,0)	(90,	90,	90,	90,	0,	0)
16	(1,1,0,0,0,0,1,0,0,0,1,0,0,0,0)	(162,	142,	142,	77,	77,	0)
17	(1,1,0,0,0,0,1,0,0,0,0,0,0,0,1)	(115,	115,	65,	65,	30,	30)
18	(1,1,0,0,0,0,0,0,0,0,0,0,1,1,0)	(80,	50,	50,	80,	50,	50)
19	(1,1,1,1,1,0,0,0,0,0,0,0,0,0,0)	(500,	260,	260,	260,	260,	260)
20	(1,1,1,1,0,1,0,0,0,0,0,0,0,0,0)	(220,	100,	100,	90,	90,	0)
21	(1,1,1,1,0,0,0,0,1,0,0,0,0,0,0)	(455,	335,	255,	255,	255,	245)
22	(1,1,1,0,0,1,1,0,0,0,0,0,0,0,0)	(95,	95,	85,	85,	0,	0)
23	(1,1,1,0,0,1,0,1,0,0,0,0,0,0,0)	(165,	165,	100,	85,	85,	0)
24	(1,1,1,0,0,1,0,0,0,0,0,0,1,0,0)	(180,	95,	95,	150,	80,	0)
25	(1,1,1,0,0,1,0,0,0,0,0,0,0,0,1)	(130,	80,	80,	70,	30,	30)
26	(1,1,1,0,0,0,0,1,1,0,0,0,0,0,0)	(400,	400,	250,	250,	250,	250)
27	(1,1,1,0,0,0,0,1,0,0,1,0,0,0,0)	(178,	113,	113,	88,	108,	0)
28	(1,1,1,0,0,0,0,1,0,0,0,1,0,0,0)	(412,	327,	327,	250,	242,	242)
29	(1,1,1,0,0,0,0,1,0,0,0,0,0,0,1)	(389,	359,	247,	247,	319,	239)
30	(1,1,0,0,0,1,0,0,0,0,0,0,1,1,0)	(60,	60,	60,	80,	50,	50)
31	(1,1,0,0,0,0,1,0,0,1,0,0,0,0,1)	(90,	90,	90,	90,	30,	30)
32	(1,1,0,0,0,0,1,0,0,0,1,0,1,0,0)	(120,	120,	120,	120,	120,	0)
33	(1,1,0,0,0,0,1,0,0,0,1,0,0,1,0)	(349,	349,	314,	314,	237,	237)
34	(1,1,1,1,1,1,0,0,0,0,0,0,0,0,0)	(480,	270,	270,	260,	260,	260)
35	(1,1,1,1,0,1,1,0,0,0,0,0,0,0,0)	(185,	115,	105,	105,	90,	0)
36	(1,1,1,1,0,1,0,0,1,0,0,0,0,0,0)	(420,	350,	270,	255,	255,	250)
37	(1,1,1,1,0,1,0,0,0,0,0,0,1,0,0)	(200,	100,	100,	100,	100,	0)
38	(1,1,1,1,0,1,0,0,0,0,0,0,0,1,0)	(435,	265,	265,	335,	255,	245)
39	(1,1,1,1,0,0,0,0,1,0,0,1,0,0,0)	(436,	286,	286,	258,	258,	276)
40	(1,1,1,0,0,1,1,0,0,1,0,0,0,0,0)	(90,	90,	90,	90,	0,	0)

Table 9: Payoffs according to the Myerson value for the game of example 5.1 and all non-isomorphic graphs, part 2.

number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
41	(1,1,1,0,0,1,1,0,0,0,1,0,0,0,0)	(118,	118,	173,	103,	88,	0)
42	(1,1,1,0,0,1,1,0,0,0,0,0,0,0,1)	(95,	95,	85,	85,	30,	30)
43	(1,1,1,0,0,1,0,1,0,0,0,1,0,0,0)	(350,	350,	350,	250,	250,	250)
44	(1,1,1,0,0,1,0,1,0,0,0,0,1,0,0)	(131,	131,	106,	116,	116,	0)
45	(1,1,1,0,0,1,0,1,0,0,0,0,0,1,0)	(377,	342,	265,	327,	247,	242)
46	(1,1,1,0,0,1,0,0,0,0,0,0,1,1,0)	(380,	260,	260,	400,	250,	250)
47	(1,1,1,0,0,1,0,0,0,0,0,0,1,0,1)	(369,	257,	257,	359,	319,	239)
48	(1,1,1,0,0,0,0,1,1,0,1,0,0,0,0)	(366,	366,	281,	253,	281,	253)
49	(1,1,1,0,0,0,0,1,0,0,1,0,1,0,0)	(132,	112,	112,	112,	132,	0)
50	(1,1,1,0,0,0,0,1,0,0,1,0,0,1,0)	(392,	280,	280,	332,	272,	244)
51	(1,1,1,0,0,0,0,1,0,0,1,0,0,0,1)	(360,	288,	288,	252,	360,	252)
52	(1,1,1,0,0,0,0,1,0,0,0,1,0,0,1)	(374,	297,	297,	254,	289,	289)
53	(1,1,0,0,0,1,0,0,0,0,0,0,1,1,1)	(60,	60,	60,	60,	60,	60)
54	(1,1,0,0,0,0,1,0,0,0,1,0,0,1,1)	(300,	300,	300,	300,	300,	300)
55	(1,1,1,1,1,1,1,0,0,0,0,0,0,0,0)	(445,	285,	275,	275,	260,	260)
56	(1,1,1,1,1,1,0,0,0,0,0,0,1,0,0)	(460,	270,	270,	270,	270,	260)
57	(1,1,1,1,0,1,1,1,0,0,0,0,0,0,0)	(135,	135,	110,	110,	110,	0)
58	(1,1,1,1,0,1,1,0,1,0,0,0,0,0,0)	(370,	370,	275,	275,	255,	255)
59	(1,1,1,1,0,1,1,0,0,1,0,0,0,0,0)	(180,	110,	110,	110,	90,	0)
60	(1,1,1,1,0,1,1,0,0,0,1,0,0,0,0)	(138,	123,	123,	108,	108,	0)
61	(1,1,1,1,0,1,1,0,0,0,0,1,0,0,0)	(373,	288,	358,	273,	255,	253)
62	(1,1,1,1,0,1,1,0,0,0,0,0,0,0,1)	(400,	280,	270,	270,	335,	245)
63	(1,1,1,1,0,1,0,0,1,0,0,1,0,0,0)	(431,	291,	291,	258,	258,	271)
64	(1,1,1,1,0,1,0,0,1,0,0,0,1,0,0)	(400,	350,	270,	265,	265,	250)
65	(1,1,1,1,0,1,0,0,1,0,0,0,0,1,0)	(389,	304,	276,	289,	258,	284)
66	(1,1,1,1,0,1,0,0,0,0,0,0,0,1,1)	(416,	268,	268,	286,	286,	276)
67	(1,1,1,1,0,0,0,0,1,0,0,1,0,1,0)	(391,	283,	283,	283,	259,	301)
68	(1,1,1,0,0,1,1,0,0,1,0,0,0,0,1)	(90,	90,	90,	90,	30,	30)
69	(1,1,1,0,0,1,1,0,0,0,1,0,1,0,0)	(117,	117,	127,	127,	112,	0)

Table 10: Payoffs according to the Myerson value for the game of example 5.1 and all non-isomorphic graphs, part 3.

number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
70	(1,1,1,0,0,1,1,0,0,0,1,0,0,1,0)	(293,	293,	355,	355,	252,	252)
71	(1,1,1,0,0,1,1,0,0,0,1,0,0,0,1)	(285,	285,	387,	267,	332,	244)
72	(1,1,1,0,0,1,0,1,0,0,0,1,1,0,0)	(308,	308,	360,	285,	285,	254)
73	(1,1,1,0,0,1,0,1,0,0,0,0,1,1,0)	(309,	301,	273,	371,	291,	255)
74	(1,1,1,0,0,1,0,1,0,0,0,0,0,1,1)	(317,	317,	274,	299,	299,	294)
75	(1,1,1,0,0,1,0,0,0,0,0,0,1,1,1)	(380,	260,	260,	380,	260,	260)
76	(1,1,1,0,0,0,0,1,1,0,1,1,0,0,0)	(368,	308,	308,	256,	280,	280)
77	(1,1,1,0,0,0,0,1,1,0,1,0,0,1,0)	(324,	324,	288,	288,	288,	288)
78	(1,1,1,0,0,0,0,1,0,0,1,0,0,1,1)	(321,	286,	286,	293,	321,	293)
79	(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0)	(395,	305,	280,	280,	280,	260)
80	(1,1,1,1,1,1,1,0,0,1,0,0,0,0,0)	(440,	280,	280,	280,	260,	260)
81	(1,1,1,1,1,1,1,0,0,0,1,0,0,0,0)	(398,	293,	293,	278,	278,	260)
82	(1,1,1,1,1,1,1,0,0,0,0,0,0,0,1)	(425,	285,	275,	275,	270,	270)
83	(1,1,1,1,0,1,1,1,0,1,0,0,0,0,0)	(130,	130,	115,	115,	110,	0)
84	(1,1,1,1,0,1,1,1,0,0,0,1,0,0,0)	(311,	311,	366,	278,	278,	256)
85	(1,1,1,1,0,1,1,0,1,1,0,0,0,0,0)	(365,	365,	280,	280,	255,	255)
86	(1,1,1,1,0,1,1,0,1,0,1,0,0,0,0)	(311,	381,	296,	278,	276,	258)
87	(1,1,1,1,0,1,1,0,1,0,0,0,0,0,1)	(327,	327,	281,	281,	292,	292)
88	(1,1,1,1,0,1,1,0,0,1,0,0,0,0,1)	(395,	275,	275,	275,	335,	245)
89	(1,1,1,1,0,1,1,0,0,0,1,0,1,0,0)	(124,	119,	119,	119,	119,	0)
90	(1,1,1,1,0,1,1,0,0,0,1,0,0,1,0)	(316,	298,	293,	363,	275,	255)
91	(1,1,1,1,0,1,1,0,0,0,0,1,0,1,0)	(375,	287,	300,	300,	258,	280)
92	(1,1,1,1,0,1,1,0,0,0,0,1,0,0,1)	(332,	296,	314,	278,	291,	289)
93	(1,1,1,1,0,1,0,0,1,0,0,1,1,0,0)	(411,	291,	291,	268,	268,	271)
94	(1,1,1,1,0,1,0,0,1,0,0,1,0,1,0)	(386,	288,	288,	283,	259,	296)
95	(1,1,1,1,0,1,0,0,1,0,0,0,1,1,0)	(332,	309,	278,	309,	278,	294)
96	(1,1,1,1,0,1,0,0,1,0,0,0,0,1,1)	(332,	304,	280,	286,	286,	312)
97	(1,1,1,1,0,0,0,0,1,0,0,1,0,1,1)	(328,	286,	286,	286,	286,	328)
98	(1,1,1,0,0,1,1,0,0,0,1,0,1,0,1)	(285,	285,	303,	303,	368,	256)

Table 11: Payoffs according to the Myerson value for the game of example 5.1 and all non-isomorphic graphs, part 4.

number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
99	(1,1,1,0,0,1,1,0,0,0,1,0,0,1,1)	(291,	291,	316,	316,	293,	293)
100	(1,1,1,0,0,1,0,1,0,0,0,1,1,1,0)	(298,	305,	305,	318,	287,	287)
101	(1,1,1,0,0,1,0,1,0,0,0,0,1,1,1)	(312,	312,	276,	312,	312,	276)
102	(1,1,1,0,0,0,0,1,1,0,1,1,1,0,0)	(308,	308,	308,	284,	308,	284)
103	(1,1,1,1,1,1,1,1,1,0,0,0,0,0,0)	(330,	330,	285,	285,	285,	285)
104	(1,1,1,1,1,1,1,1,0,1,0,0,0,0,0)	(390,	300,	285,	285,	280,	260)
105	(1,1,1,1,1,1,1,1,0,0,0,1,0,0,0)	(336,	316,	301,	283,	283,	281)
106	(1,1,1,1,1,1,1,0,0,1,0,0,0,0,1)	(420,	280,	280,	280,	270,	270)
107	(1,1,1,1,1,1,1,0,0,0,1,0,1,0,0)	(384,	289,	289,	289,	289,	260)
108	(1,1,1,1,1,1,1,0,0,0,1,0,0,1,0)	(341,	303,	298,	298,	280,	280)
109	(1,1,1,1,0,1,1,1,0,1,1,0,0,0,0)	(122,	122,	122,	117,	117,	0)
110	(1,1,1,1,0,1,1,1,0,1,0,1,0,0,0)	(303,	303,	373,	285,	278,	258)
111	(1,1,1,1,0,1,1,1,0,1,0,0,0,0,1)	(306,	306,	283,	283,	366,	256)
112	(1,1,1,1,0,1,1,1,0,0,0,1,0,1,0)	(311,	311,	306,	306,	282,	284)
113	(1,1,1,1,0,1,1,0,1,1,0,0,0,0,1)	(322,	322,	286,	286,	292,	292)
114	(1,1,1,1,0,1,1,0,1,0,1,1,0,0,0)	(319,	319,	319,	281,	281,	281)
115	(1,1,1,1,0,1,1,0,1,0,1,0,1,0,0)	(295,	378,	290,	290,	288,	259)
116	(1,1,1,1,0,1,1,0,1,0,1,0,0,1,0)	(322,	322,	299,	299,	279,	279)
117	(1,1,1,1,0,1,1,0,1,0,1,0,0,0,1)	(311,	324,	293,	282,	304,	286)
118	(1,1,1,1,0,1,1,0,0,0,1,0,0,1,1)	(306,	295,	295,	308,	308,	288)
119	(1,1,1,1,0,1,1,0,0,0,0,1,0,1,1)	(316,	292,	298,	298,	287,	309)
120	(1,1,1,1,0,1,0,0,1,0,0,1,1,1,0)	(327,	291,	291,	304,	280,	307)
121	(1,1,1,1,0,1,0,0,1,0,0,1,0,1,1)	(323,	291,	291,	286,	286,	323)
122	(1,1,1,0,0,1,0,1,0,0,0,1,1,1,1)	(300,	300,	300,	300,	300,	300)
123	(1,1,1,0,0,0,0,1,1,0,1,1,1,1,0)	(300,	300,	300,	300,	300,	300)
124	(1,1,1,1,1,1,1,1,1,1,0,0,0,0,0)	(325,	325,	290,	290,	285,	285)
125	(1,1,1,1,1,1,1,1,0,1,1,0,0,0,0)	(382,	292,	292,	287,	287,	260)
126	(1,1,1,1,1,1,1,1,0,1,0,1,0,0,0)	(328,	308,	308,	290,	283,	283)
127	(1,1,1,1,1,1,1,1,0,1,0,0,0,0,1)	(331,	311,	288,	288,	301,	281)

Table 12: Payoffs according to the Myerson value for the game of example 5.1 and all non-isomorphic graphs, part 5.

number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
128	(1,1,1,1,1,1,1,1,0,0,0,1,0,1,0)	(320,	313,	295,	295,	284,	293)
129	(1,1,1,1,1,1,1,0,0,0,1,0,0,1,1)	(315,	297,	297,	297,	297,	297)
130	(1,1,1,1,0,1,1,1,0,1,1,0,1,0,0)	(120,	120,	120,	120,	120,	0)
131	(1,1,1,1,0,1,1,1,0,1,1,0,0,1,0)	(293,	293,	293,	376,	286,	259)
132	(1,1,1,1,0,1,1,1,0,1,0,1,0,1,0)	(309,	309,	309,	309,	282,	282)
133	(1,1,1,1,0,1,1,1,0,1,0,1,0,0,1)	(301,	301,	314,	290,	307,	287)
134	(1,1,1,1,0,1,1,1,0,0,0,1,0,1,1)	(303,	303,	298,	298,	298,	300)
135	(1,1,1,1,0,1,1,0,1,0,1,1,1,0,0)	(301,	314,	314,	294,	294,	283)
136	(1,1,1,1,0,1,1,0,1,0,1,0,1,0,1)	(298,	315,	293,	293,	315,	286)
137	(1,1,1,1,0,1,1,0,1,0,1,0,0,1,1)	(306,	306,	295,	295,	299,	299)
138	(1,1,1,1,0,1,0,0,1,0,0,1,1,1,1)	(318,	291,	291,	291,	291,	318)
139	(1,1,1,1,1,1,1,1,1,1,1,0,0,0,0)	(317,	317,	297,	292,	292,	285)
140	(1,1,1,1,1,1,1,1,1,1,0,0,0,0,1)	(320,	320,	290,	290,	290,	290)
141	(1,1,1,1,1,1,1,1,0,1,1,0,1,0,0)	(380,	290,	290,	290,	290,	260)
142	(1,1,1,1,1,1,1,1,0,1,1,0,0,1,0)	(318,	298,	298,	311,	291,	284)
143	(1,1,1,1,1,1,1,1,0,1,0,1,0,0,1)	(310,	303,	303,	292,	296,	296)
144	(1,1,1,1,1,1,1,1,0,0,0,1,0,1,1)	(307,	304,	295,	295,	295,	304)
145	(1,1,1,1,0,1,1,1,0,1,1,0,0,1,1)	(296,	296,	296,	313,	313,	286)
146	(1,1,1,1,0,1,1,1,0,1,0,1,0,1,1)	(301,	301,	301,	301,	298,	298)
147	(1,1,1,1,0,1,1,0,1,0,1,1,1,1,0)	(301,	301,	305,	305,	294,	294)
148	(1,1,1,1,1,1,1,1,1,1,1,1,0,0,0)	(306,	306,	306,	294,	294,	294)
149	(1,1,1,1,1,1,1,1,1,1,1,0,1,0,0)	(315,	315,	295,	295,	295,	285)
150	(1,1,1,1,1,1,1,1,1,1,1,0,0,1,0)	(307,	307,	300,	300,	293,	293)
151	(1,1,1,1,1,1,1,1,0,1,1,0,0,1,1)	(305,	298,	298,	302,	302,	295)
152	(1,1,1,1,0,1,1,0,1,0,1,1,1,1,1)	(300,	300,	300,	300,	300,	300)
153	(1,1,1,1,1,1,1,1,1,1,1,1,1,0,0)	(304,	304,	304,	297,	297,	294)
154	(1,1,1,1,1,1,1,1,1,1,1,0,0,1,1)	(302,	302,	299,	299,	299,	299)
155	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)	(301,	301,	301,	301,	298,	298)
156	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	(300,	300,	300,	300,	300,	300)

Table 13: Payoffs according to the Myerson value for the game of example 5.1 and all non-isomorphic graphs, part 6.